THE ELEMENTS

OF

HYDROSTATICS

AND

HYDRODYNAMICS.

By W. H. MILLER, M.A.

FELLOW AND TUTOR OF ST. JOHN'S COLLEGE, AND PROFESSOR OF
MINERALOGY IN THE UNIVERSITY OF CAMBRIDGE.

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CONTENTS.

SECTION I.

SECTION II.

CONTENTS.

SECTION I.

SECTION II.

CONTENTS.

SECTION I.

SECTION II.

SECTION III.

SECTION IV.

SECTION V.

SECTION VI.

SECTION VII.

ELEMENTS OF HYDROSTATICS.

SECTION I.

GENERAL PROPERTIES OF FLUIDS.

Art. 1. A fluid is a body which can be divided in any direction, and the parts of which can be moved among one another by any assignable force.

Elastic fluids are those the dimensions of which are increased or diminished when the pressure upon them is diminished or increased. Non-elastic fluids are those the dimensions of which are independent of the pressure.

Water, mercury, and probably all other liquids, are in a small degree compressible. Their resistance however to compression is so great, that the conclusions obtained on the supposition of their being incompressible, are in most cases free from any sensible error.

2. Let DEF (fig. 1) be a piston without weight exactly fitting an orifice in the plane ABC, which forms the side of a vessel containing fluid. It is manifest that the fluid can make no effort to move the piston in any other direction than that of a normal to its surface, the piston may therefore be kept at rest by a force applied at some point G in it, and acting in a direction HG perpendicular to DEF. A force equal and opposite to this is called the pressure of the fluid on DEF.

The pressure of a fluid at a given point is measured by the quantity p, $p\kappa$ being the pressure of the fluid on an indefinitely small area κ contiguous to the given point.

When the pressure of a fluid on a given surface is the same, wherever that surface is placed, p is the pressure on a unit of surface. When the pressure on a given surface, varies with the situation of the surface, p is the pressure which would be exerted on a unit of surface, if the pressure at each part of the unit of surface were equal to the pressure at the given point.

3. The characteristic property of fluids, which distinguishes them essentially from solids, is the faculty they possess of transmitting equally, and in all directions, the pressures applied to their surfaces. This property, which forms the basis of the theory of the equilibrium of fluids, is sometimes assumed as self-evident; it admits however of the following demonstration, founded upon the principle, that when a fluid is at rest, any portion of it may become solid without disturbing its own equilibrium, or disturbing the equilibrium or altering the pressure of the surrounding fluid.

To prove that fluids press equally in all directions.

Let the fluid contained within the prism Abc (fig. 2), in the interior of a fluid at rest, become solid. The equilibrium of Abc and the pressure of the surrounding fluid will not be altered. Let R be the accelerating force at A, and therefore R.(mass prism) the moving force on the prism: the only other forces that act upon it are the pressures upon its ends and sides. Since the prism is at rest, the forces, R.(mass prism), and the pressures upon the ends and sides must be in equilibrium. If the prism be diminished indefinitely, retaining its original proportions, the force R.(mass prism) will vanish compared with the pressures of the surrounding fluid, for the former is proportional to Aa^3, the latter to Aa^2, and we may consider the prism to be kept at rest by the pressures upon its ends and sides. These pressures are respectively parallel and perpendicular to ABC, therefore they must be separately in equilibrium. Since the pressures on Ab, Ac, Cb, are in equilibrium and perpendicular to the sides AB, AC, CB of the triangle ABC, they are proportional to those sides; hence, if $p.Ab$, $q.Ac$ be the pressures upon the sides Ab, Ac respectively,

$$\frac{p.Ab}{q.Ac} = \frac{AB}{AC}, \text{ therefore, since } \frac{Ab}{Ac} = \frac{AB}{AC}, \; p = q.$$

But (2) p, q measure the pressures of the fluid at A perpendicular to Ab, Ac respectively, and Ab, Ac may be taken perpendicular to any two given lines. Hence fluids press equally in all directions.

4. Suppose the sides of the base of the prism to be indefinitely small compared with its length. Then, if the pressure on ABC be increased or diminished in any degree without disturbing the equilibrium of Abc, the pressure on abc must be equally increased or diminished. Hence if F, G, H...... M, N, P (fig. 3) be any series of points in a fluid at rest, so taken that the straight lines FG, GH......MN, NP may be wholly within the fluid, and the pressure at F be increased or diminished without disturbing the equilibrium of the fluid, the pressures at G, H......M, N, P will be equally increased or diminished.

If the fluid be acted on by no accelerating force, the pressures on ABC, abc must be equal; therefore pressure at F = pressure at G = ... = pressure at N = pressure at P: or, the pressure is the same at all points in a fluid at rest acted on by no accelerating force.

5. Let A, B, C,...be pistons fitting cylindrical pipes which communicate with the inside of a vessel filled with fluid; and let the forces P, Q, R,...be in equilibrium when applied to the pistons A, B, C,...in directions parallel to the axes of the cylinders and tending inwards. Let a, b, c be the areas of the pistons, and suppose the fluid to be acted upon by no accelerating force, then, the fluid being at rest, the pressures which it exerts upon a unit of the surface of each piston must be equal, therefore (2)

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} = \ldots\ldots$$

6. Let the fluid be incompressible; p, q, r,...the distances of the pistons from fixed points in the axes of the cylinders in which they play and without the fluid; p', q', r',... their distances from the same points after they have been moved in any manner so that they still remain in contact with

the fluid. Let V be the volume of the portion of the vessel bounded by sections passing through the fixed points perpendicular to the axes of the cylinders, then, the volume of the fluid at first will be

$$V - (ap + bq + cr + \ldots\ldots),$$

and after the pistons have been moved

$$V - (ap' + bq' + cr' + \ldots\ldots).$$

These are equal, for the whole quantity of fluid remains unaltered, therefore

$$a(p' - p) + b(q' - q) + c(r' - r) + \ldots\ldots = 0.$$

$$\text{But } \frac{P}{a} = \frac{Q}{b} = \frac{R}{c} = \ldots\ldots \ (5);$$

$$\therefore\ P(p' - p) + Q(q' - q) + R(r' - r) + \ldots\ldots = 0.$$

$p' - p$, $q' - q$, $r' - r$, ... the spaces described by the pistons estimated in the directions in which the forces act, are proportional to the virtual velocities of the pistons. Hence, the sum of the products of each force into the virtual velocity of the piston to which it is applied $= 0$.

SECTION II.

ON THE EQUILIBRIUM OF NON-ELASTIC FLUIDS ACTED ON BY GRAVITY.

ART. 7. CERTAIN standard units of space, volume and weight having been assumed, the specific gravity, mass and density of a body may be defined as follows.

The specific gravity of a body is the weight of a unit of its volume.

The density of a body is the quantity of matter in a unit of its volume.

Let W, M, V be the number of units of weight, mass, and volume contained in the weight, mass, and volume of a given body, S its specific gravity, D its density, g the force of gravity; then

$$S = \text{weight of one unit} = gD,$$

$$M = \text{mass of } V \text{ units} = DV,$$

$$W = \text{weight of } V \text{ units} = SV = gDV.$$

8. When a fluid acted on by gravity is at rest, the pressures are equal at all points in the same horizontal plane.

Let P, Q (fig. 4) be any two points in the same horizontal plane in the interior of a fluid at rest, and let the fluid contained within a very slender prism, of which PQ is one edge, become solid. The prism is horizontal, therefore the pressures upon its ends are the only forces that act upon it in the direction PQ; and it remains at rest, therefore these pressures are equal. But the ends are equal, therefore the pressures at P and Q are equal.

If any portion of the fluid become solid without interrupting the communication by means of a canal of any form between P and Q, the pressures at those points will remain unchanged. Hence the pressure of the fluid is the same at all points in the same horizontal plane, whatever be the form of the vessel containing it.

9. To find the pressure at any point in a mass of fluid at rest.

Let the fluid contained within the vertical prism AEF (fig. 5), in the interior of a fluid at rest, become solid. Its weight, and the pressures upon its ends, are the only forces that act upon it in a vertical direction, therefore, since the prism is at rest, these must be in equilibrium; therefore pressure on DEF = weight of prism AEF + pressure on ABC. Let m, p be the pressures at A, D respectively, ρ the density of the fluid, $ABC = \kappa$, $AD = z$; then the pressure on $ABC = m\kappa$, the pressure on $DEF = p\kappa$, and the weight of $AEF = g\rho z\kappa$; therefore $p\kappa = g\rho z\kappa + m\kappa$, therefore

$$p = g\rho z + m.$$

When A is a point in the open surface of the fluid, having no other fluid above it, $m = 0$, and therefore $p = g\rho z$.

Since fluids press equally in all directions, and the pressure is the same at all points in the same horizontal plane, the pressure on a small area of any plane is ultimately equal to the pressure on an equal area of the horizontal plane that intersects it.

10. The surface of a fluid at rest is a horizontal plane.

Let A, P (fig. 6) be any two points in the surface of a fluid at rest, AB, PQ vertical straight lines intersected by a horizontal plane in B, Q; ρ the density of the fluid. Then, (8, 9) $g\rho.PQ$ = pressure at Q = pressure at $B = g\rho.AB$; therefore $PQ = AB$, therefore A and P are in the same horizontal plane.

11. The common surface of two fluids that do not mix is a horizontal plane.

Let A, P (fig. 7) be any two points in the common surface of two fluids that do not mix; BAC, QPR vertical straight lines intersected by horizontal planes in B, Q, and in C, R; ρ, σ the densities of the upper and under fluids respectively. Then, (9) pressure at A − pressure at $B = g\rho . AB$,

$$\text{also pressure at } C - \text{pressure at } A = g\sigma . AC,$$

$$\therefore \text{ pressure at } C - \text{pressure at } B = g . (\rho . AB + \sigma . AC);$$

in like manner

$$\text{pressure at } R - \text{pressure at } Q = g(\rho . PQ + \sigma . PR),$$

$$\text{and, (8) pres. at } Q = \text{pres. at } B, \text{ pres. at } R = \text{pres. at } C,$$

$$\therefore \rho . PQ + \sigma . PR = \rho . AB + \sigma . AC,$$

$$\text{and } \sigma . PQ + \sigma . PR = \sigma . AB + \sigma . AC,$$

$$\therefore (\sigma - \rho) . PQ = (\sigma - \rho) . AB;$$

$\therefore$ $PQ = AB$, $\therefore$ A and P are in the same horizontal plane.

12. If two fluids that do not mix meet in a bent tube, the altitudes of their surfaces above the horizontal plane in which they meet are inversely as their densities.

Let PAQ (fig. 8) be a bent tube containing two fluids of different densities; AP, AQ the portions of the tube occupied by the lighter and heavier fluids; ρ, σ the densities of the fluids in AP, AQ. Let the planes of the surfaces of the fluids, and the plane in which they meet cut a vertical in H, K, C.

The pressure of the fluid in AP at $A = g\rho . HC$,

and the pressure of the fluid in AQ at $A = g\sigma . KC$.

When the fluids are in equilibrium these pressures must be equal, therefore $\rho . HC = \sigma . KC$;

$$\therefore \frac{\rho}{\sigma} = \frac{KC}{HC}.$$

13. To find the pressure of a fluid on any surface.

Let BPC (fig. 9) be the given surface. Draw AK vertical meeting the surface of the fluid in A, through H, K draw horizontal planes meeting the surface BPC in the curves PM, QN. Let P be the pressure on MPB, S the area of MPB, ρ the density of the fluid, X the depth of the center of gravity of BPC below the surface of the fluid, $AH = x$, $HK = \delta x$, therefore, ultimately, pressure on $MQ = d_x P . \delta x$, area $MQ = d_x S . \delta x$. But, ultimately,

$$\text{pressure on } MQ = g\rho . AH . MQ = g\rho . x . d_x S . \delta x ;$$

$$\therefore \; d_x P = g\rho . x . d_x S ;$$

$$\therefore \; P = g\rho . \int_x x . d_x S ,$$

and the pressure on the whole surface BPC

$$= g\rho . \int_x x . d_x S ,$$

the integral being taken between the limits corresponding to the highest and lowest points in the surface.

But $X . (\text{area } BPC) = \int_x x . d_x S$ between the same limits;

$$\therefore \; \text{pressure on } BPC = g\rho X . (\text{area } BPC) ;$$

or, the pressure of a fluid on any surface is equal to the weight of a column of the fluid the base of which is equal to the area of the surface, and altitude equal to the depth of the center of gravity of the surface below the surface of the fluid.

When the surface BPC is a plane, the pressures are all perpendicular to BPC, and consequently parallel to each other; therefore the resultant of the pressure on BPC is equal to the whole pressure, and acts in a direction perpendicular to BPC.

14. The centre of pressure of a plane surface immersed in a fluid is the point in which the resultant of the pressure of the fluids meets that surface.

To find the center of pressure of any plane surface.

Let ABC (fig. 10) be the surface, OY the line in which its plane cuts the surface of the fluid. From O draw OX in the plane ABC perpendicular to OY, and let X, Y be the co-ordinates of the center of pressure referred to the axes OX, OY.

Then, since the pressures are parallel to each other, we shall have, (Whewell's Mechanics, 84; Snowball's Mechanics, 169).

$X . (\text{pressure on } ABC) = \text{moment of pressure on } ABC \text{ round } OY$,

$Y . (\text{pressure on } ABC) = \text{moment of pressure on } ABC \text{ round } OX$.

Draw MP, NQ parallel to OX; HP, KQ parallel to OY; PT perpendicular to the surface of the fluid meeting it in T.

Let $OH = x$, $HK = \delta x$, $OM = y$, $MN = \delta y$, $TMP = \theta$, the density of the fluid $= \rho$. Therefore, ultimately, pressure on $PQ = g\rho . PT . PQ = g\rho . \sin\theta . x . \delta x . \delta y$;

moment of the pressure on PQ round OY

$$= g\rho . MP . PT . PQ = g\rho . \sin\theta . x^2 . \delta x . \delta y;$$

moment of the pressure on PQ round OX

$$= g\rho . HP . PT . PQ = g\rho . \sin\theta . xy . \delta x . \delta y;$$

$$\therefore \text{ pressure on } ABC = g\rho . \sin\theta . \int_x \int_y x;$$

moment of the press. on ABC round $OY = g\rho . \sin\theta . \int_x \int_y x^2$;

moment of the press. on ABC round $OX = g\rho . \sin\theta . \int_x \int_y xy$;

the integrals being taken between limits corresponding to the boundary of the surface.

$$\therefore X . \int_x \int_y x = \int_x \int_y x^2, \quad Y . \int_x \int_y x = \int_x \int_y xy.$$

If ABC were a plane lamina of very small uniform thickness, moveable round the axis OY, the above values of X and Y would be those of the co-ordinates of its "center of percussion."

15. Let one side of the plane ABC be exposed to the pressure of the fluid; then, since the center of pressure of ABC is the point of application of the resultant of the pressure of the fluid on ABC, the plane may be kept at rest by a single force equal and opposite to the pressure of the fluid acting in a perpendicular to it through its center of pressure.

16. To find the vertical pressure of a fluid on any surface.

Let PQR, (fig. 11) be the given surface; let the surface generated by a vertical line moving along the boundary of PQR meet the surface of the fluid in ABC; and suppose the fluid within ABR to become solid. The vertical pressure of the fluid on PQR and the weight of ABR are the only forces that act vertically on ABR; therefore, since ABR remains at rest, these forces must be equal to each other, and act in the same straight line, in opposite directions. Hence the pressure of the

CONTENTS.

SECTION I.

SECTION II.

out altering the pressure at any point in V. In this case the pressure at any point in the surface of V will be equal and opposite to the pressure at the same point in the former case. Consequently the resultant of the pressure in the latter case will be equal and opposite to the resultant of the pressure in the former case. Hence the resultant of the pressure of a fluid on the inside of the vessel containing it is equal to the weight of the fluid, and acts downwards in a vertical through the center of gravity of the fluid.

21. When a solid floats in equilibrium, the weight of the solid is equal to the weight of the fluid displaced, and the line joining the centers of gravity of the solid and of the fluid displaced is vertical.

The weight of the solid and the pressure of the fluid on the surface of the portion of the solid immersed are the only forces that act upon the solid. Therefore, since the solid is at rest, the weight of the solid and the resultant of the pressure of the fluid on its surface must act in opposite directions in the same straight line. But the weight of the solid acts downwards in a vertical through the center of gravity of the solid, and the resultant of the pressure of the fluid is equal to the weight of the fluid displaced, and acts upwards in a vertical through the center of gravity of the fluid displaced. Hence the weight of the solid is equal to the weight of the fluid displaced, and the line joining the centers of gravity of the solid and of the fluid displaced is vertical.

22. To find the conditions of equilibrium of a solid suspended in a fluid by a string.

Let GN, FM (fig. 13) be verticals through the centers of gravity of the solid, and of the fluid displaced by it, EL the direction of the string by which the solid is suspended, T the tension of the string, W the weight of the solid, V the volume of the fluid displaced, ρ its density, and therefore $g\rho V$ the weight of the fluid displaced, or (21) the resultant of the pressure of the fluid on the solid. Now W acts downwards in GN, $g\rho V$ acts upwards in FM; hence in order that the solid may be kept at rest by T acting in EL, EL must be vertical, and in the same plane with FM, GN; $T = W - g\rho V$ acting upwards, or $g\rho V - W$ acting downwards, according as W is greater or less than $g\rho V$; and if EGF be drawn perpendicular to GN in the plane GFN, $W.GE = g\rho V.FE$.

23. W acting downwards in GN, and $g\rho V$ acting upwards in FM may be resolved into a single force $W - g\rho V$ acting downwards in GN, and a "couple" $g\rho V.FG$ in the plane MGF tending to make the solid revolve in the direction GFM; hence if any forces acting on the solid can be resolved into a single force $W - g\rho V$ acting upwards in NG, and a "couple" $g\rho V.FG$ in the plane MGF tending to make the solid revolve in the direction MGF, they will keep it at rest.

24. To find the positions in which a solid can float in equilibrium.

Let $f(x, y, z) = 0$ be the equation to the surface of the solid, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ the equation to the surface of the fluid, the center of gravity of the solid being the origin of the co-ordinates; V the volume of the fluid displaced by the solid; X, Y, Z the co-ordinates of the center of gravity of the fluid displaced; ρ the density of the fluid; W the weight of the solid. Then $g\rho V$ will be the weight of the fluid displaced, and $\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z}$ the equations to the line joining the centers of gravity of the solid and of the fluid displaced. But when the solid is at rest, its weight is equal to the weight of the fluid displaced, and the line joining the centers of gravity of the solid and of the fluid displaced is perpendicular to the surface of the fluid, therefore

$$W = g\rho V, \text{ and } aX = bY = cZ. \quad \text{Also}$$

$$V = \int_x \int_y \int_z 1, \quad V.X = \int_x \int_y \int_z x, \quad V.Y = \int_x \int_y \int_z y, \quad V.Z = \int_x \int_y \int_z z.$$

The limits of the integrations being determined by the equations

$$f(x, y, z,) = 0, \text{ and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Whence, having found the different values of a, b, c, the equation to the surface of the fluid corresponding to each position of equilibrium of the solid becomes known.

The section of a solid floating in equilibrium made by the surface of the fluid, is called the plane of floatation.

25. To determine whether the equilibrium of a floating solid is stable or unstable.

Let the equilibrium of the solid be slightly disturbed by making it revolve through a very small angle in a vertical plane, without altering the quantity of fluid displaced. The resultant of the pressure of the fluid on the solid in its new position will still be equal to the weight of the solid, and will therefore have no tendency to elevate or depress the center of gravity of the solid; but, since it acts in a vertical through the center of gravity of the fluid displaced, unless the line joining the centers of gravity of the solid and of the fluid displaced should happen to be vertical, will tend to make the solid rotate round a horizontal axis through its center of gravity. When a vertical plane can be found, in which the solid being slightly inclined, the pressure of the fluid tends to make the solid recede farther from its original position, that is, when a force acting upwards in a vertical through the center of gravity of the fluid displaced tends to increase the angle through which the solid has revolved, the equilibrium of the solid will be unstable. When on the contrary the pressure of the fluid tends to diminish the angle through which the solid has revolved, in whatever vertical plane it may have been inclined, the equilibrium will be stable.

26. When the equilibrium of a solid is slightly disturbed by making it revolve through a very small angle in a given vertical plane; to find the vertical through the center of gravity of the fluid displaced, the plane of floatation being symmetrical with respect to the vertical plane through the center of gravity of the solid in which the solid has been inclined.

Let G, H (fig. 15) be the centers of gravity of the solid and of the fluid displaced by it, when floating in equilibrium. Let a plane through GH meet the plane of floatation in ACB, and the surface of the solid in ADB. Suppose the solid to revolve through a very small angle θ in the plane ADB, so that the quantity of fluid displaced may be the same as before; and let ADB meet the surface of the fluid in aCb. Draw MF vertical through the center of gravity of the fluid displaced by the solid in its new position, and mp, nq vertical through the centers of gravity of the wedges ACa, BCb. Then since the plane of floatation is symmetrical with respect to the plane ADB, mp, nq, and consequently MF will be in the plane ADB. Draw HFE parallel to ab.

Now, if a body be divided into any number of parts, the moment of the whole body with respect to a given plane is equal to the sum of the moments of each part with respect to the same plane. Hence, since the density is uniform and therefore the mass proportional to the volume, (vol. aDb).FE + (wedge ACa).Cm = moment of $ACbD$ with respect to a plane through CE perpendicular to ABD = (vol. ADB) . HE − (wedge BCd).Cn.

Let x, y be the co-ordinates of any point in the plane of floatation, ACB being the axis of x, and YCY' the intersection of the surface of the fluid and the plane of floatation, and therefore a perpendicular to ACa, the axis of y. If vertical planes be drawn parallel to CY at the distances x, $x + \delta x$, the portion of either wedge contained between them will be $2\theta xy\delta x$, and the moment of this portion round CY will be $2\theta x^2 y \delta x$. Hence

$$\text{wedge } ACa = 2\theta \int_x xy, \text{ from } C \text{ to } A,$$
$$= \theta\,(\text{moment of } YAY' \text{ round } CY);$$
$$(\text{wedge } ACa)\ Cm = 2\theta \int_x x^2 y, \text{ from } C \text{ to } A$$
$$= \theta\,(\text{mom. inert. of } YAY' \text{ round } CY).$$

In like manner

$$\text{wedge } BCb = \theta\,(\text{moment of } YBY' \text{ round } CY),$$
$$(\text{wedge } BCb)\ Cn = \theta\,(\text{mom. inert. of } YBY' \text{ round } CY).$$

The volumes aDb, ADB are equal; therefore, subtracting aDB from each,

$$\text{wedge } ACa = \text{wedge } BCb;$$

hence the moments of YAY' and YBY' round CY are equal, and therefore CY passes through the center of gravity of the plane of floatation.

$$\begin{aligned}(\text{wedge } ACa)\, Cm + (\text{wedge } BCb)\, Cn &= \theta\, (\text{mom. inert. } YAY' \text{ round } CY)\\ &+ \theta\, (\text{mom. inert. } YBY' \text{ round } CY)\\ &= \theta k^2 A,\end{aligned}$$

where $k^2 A$ is the moment of inertia of the plane of floatation round CY. If the volume of the fluid displaced $= V$, we have

$$(\text{vol. } ADB) \,.\, HE - (\text{vol. } aDb) \,.\, FE = V.HF = V.HM.\theta;$$

$$\therefore V.HM = k^2 A.$$

The point M, in which FM ultimately cuts HG is called the metacentre.

A force acting in the direction FM will tend to diminish or increase the angle HMF according as M is above or below G; therefore the equilibrium of the solid will be stable or unstable according as M is above or below G.

27. If the plane of floatation be not symmetrical with respect to ADB, let aYb (fig. 16) be the section of the solid made by the surface of the fluid; and let H, G, &c. be the projections of H, G, &c. in (fig. 15) on the plane aYb. Draw pr, qs, MN perpendicular to ab. It may be proved as before, that the center of gravity of the plane of floatation lies in CY, and that $V.HN = \theta k^2 A$.

Also

$$V.MN + (\text{wedge } YaY').pr = (\text{wedge } YbY').qs.$$

If two vertical planes be drawn parallel to CY at the distances x, $x + \delta x$, and if two other vertical planes be drawn parallel to ab at the distances y, $y + \delta y$, the portion of either wedge contained between them will be $\theta x \delta x \delta y$, and the moment of this portion round ab will be $\theta x y \delta x \delta y$.

$$(\text{wedge } YaY').pr = -\theta.\textstyle\int_x\int_y xy, \text{ from } a \text{ to } C;$$

$$(\text{wedge } YbY').qs = \theta.\textstyle\int_x\int_y xy, \text{ from } C \text{ to } b;$$

$$\therefore (\text{wedge } YbY').qs - (\text{wedge } YaY').pr$$

$$= \theta.\textstyle\int_x\int_y xy \text{ from } C \text{ to } b + \theta.\int_x\int_y xy \text{ from } a \text{ to } C$$

$$= \theta.\textstyle\int_x\int_y xy, \text{ from } a \text{ to } b;$$

$$\therefore \text{ if } T = \textstyle\int_x\int_y xy \text{ from } a \text{ to } b,$$

$$V.MN = \theta T.$$

The equilibrium will be stable or unstable according as H and G lie on the same or on opposite sides of MN.

SECTION III.

ON THE EQUILIBRIUM OF ELASTIC FLUIDS ACTED ON BY GRAVITY.

ART. 28. To measure the pressure of the atmosphere.

Let a glass tube ABC (fig. 17) closed at the end A, be bent at B and D, so that the branches AD, BC may have a common axis PQ, AB being about thirty one inches longer than BC. Then if AB be filled with mercury, and placed so that PQ may be vertical, the mercury will sink in AB, leaving a vacuum in the upper part of the tube, and rise in BC till the pressure of the mercury at the common surface of the air and mercury in BC is equal to the pressure of the atmosphere. Let PQ meet the upper and lower surfaces of the mercury in P, Q; and let Π be the pressure of the atmosphere, σ the density of the mercury; then (9) the pressure of the mercury at $Q = g\sigma . PQ$; and this must be equal to the pressure of the atmosphere at Q when the mercury is at rest;

$$\therefore\ \Pi = g\sigma . PQ.$$

An instrument of this description furnished with a scale for measuring PQ, is called a barometer.

29. If σ_0, σ_t be the densities of mercury at 0°, t° (Centigrade) as indicated by a mercurial thermometer (see Sect. VII.) it is found that $\sigma_t(1 + et) = \sigma_0$, where $e = 0{,}00018018$.

Whence $\sigma_t = \sigma_0(1 - et)$ very nearly.

If in fig. 17 $PQ = h$, and if the temperature of the mercury $= t$,

$$\Pi = g\sigma_t h = g\sigma_0 h(1 - et).$$

30. The mean pressure of the atmosphere at the level of the sea appears to vary with the latitude. The heights of the column of mercury which it supports in different latitudes, according to the most trustworthy observations, are as follows,

the temperature of the mercury being that of melting snow, and the heights expressed in English inches:

Lat.	Height.	Lat.	Height.
0^0	29,930	49^0	29,978
10	29,975	$51\frac{1}{2}$	29,951
20	30,064	$54\frac{1}{2}$	29,926
30	30,108	60	29,803
40	30,019	64	29,606
45	30,000	67	29,673.

The mean pressure of the atmosphere at the level of the sea is greater at about 9 A.M. and 9 P.M., and is less at 3 P.M. and 3 A.M. than the mean pressure for the whole day. The difference between the greatest and least of these pressures, or the diurnal oscillation, as it is usually called, is equivalent to the pressure of a column of mercury the height of which is expressed in inches by the formula

$$0,1193\,(\cos \text{lat.})^{\frac{5}{2}} - 0,0149.$$

(Report on Meteorology, by Prof. Forbes.) The mean pressure is also subject to an annual oscillation, the magnitude of which, except for some particular spots, has not yet been ascertained. Within a zone which extends probably to the parallel of 40^0 on either side of the equator the greatest and least atmospheric pressures appear to correspond to the greatest and least meridian zenith distances of the sun. Thus at Madras (lat. $13^0\,4'$ N.) the mean height of the mercury in the barometer in January is 0,21 inches greater than in July. At Calcutta (lat. $22\frac{1}{2}^0$ N.) the difference amounts to 0,52 inches. At the Cape of Good Hope (lat. 34^0 S.) the height is 0,29 greater in July than in January.

31. The pressure of air at a given temperature varies inversely as the space it occupies.

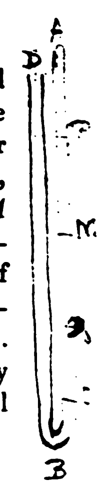
Let a glass tube *ABD* (fig. 18) closed at *A*, and having the shorter branch *AB* bent parallel to the longer *BD*, be placed so that *PC* the axis of *AB* may be vertical. Pour a small quantity of mercury into *BD*, and by withdrawing some of the air in AB or adding mercury make it stand at the same height in both branches. Let its surface meet PC in P. Pour in more mercury and let the surface of the mercury in AB and a horizontal plane touching the surface of the mercury in BD meet PC in M and C respectively. Then, if the ratio of the spaces AP, AM successively occupied by the air be measured, which may be done by weighing the quantities of mercury they respectively contain, and if h be the altitude of the mercury in the barometer at the time of making the experiment, it will be found that

$$\frac{h + MC}{h} = \frac{\text{vol.}\,AP}{\text{vol.}\,AM}.$$

But if Π, M be the pressures of the air in AB when occupying the spaces AP, AM, and σ the density of the mercury,

$$\Pi = \text{pressure of exterior air} = g\sigma h,\quad \text{M} = g\sigma h + g\sigma . MC;$$

$$\therefore \frac{\text{M}}{\Pi} = \frac{\text{vol.}\,AP}{\text{vol.}\,AM}.$$



Next, let a glass tube ABD (fig. 19) closed at A and having the branches AB, BD parallel and nearly equal, be placed so that PC, the axis of AB, may be vertical. Pour mercury into BD, withdraw any portion of the air in AB, and add mercury till the surfaces of the mercury in DB, BA stand at the same height, the latter meeting PC in P. Withdraw a portion of the mercury in BD, and let the surface of the mercury in AB, and a horizontal plane touching the surface of the mercury in BD, meet PC in M and C respectively. Then, as in the former case, if the spaces AP, AM successively occupied by the air be measured, as well as h and MC, it will be found that

$$\frac{h - MC}{h} = \frac{\text{vol.}\,AP}{\text{vol.}\,AM}.$$

But if Π, M be the pressures of the air in AB when occupying the spaces AP, AM,

$$\Pi = g\sigma h,\quad \text{M} = g\sigma h - g\sigma . MC;$$

$$\therefore \frac{\text{M}}{\Pi} = \frac{\text{vol.}\,AP}{\text{vol.}\,AM}.$$

Hence the pressure of air at a given temperature varies inversely as the space it occupies, when the pressure is less than that of the atmosphere, as well as when it is greater.

This law was first discovered by Boyle in 1662.

32. The pressure of atmospheric air at a given temperature varies inversely as the space it occupies, and therefore directly as its density. Hence, if ρ_0 be the density of atmospheric air at $0^0 C$ under the pressure Π, $\Pi = \mu\rho_0$, where μ is constant.

If h be the altitude of the column of mercury supported by the pressure of the air, σ its density, and g the force of gravity, $\Pi = g\sigma h$, therefore $g\sigma h = \mu\rho_0$.

The experiments of MM. Biot and Arago, combined with those of MM. Dumas and Boussingault (Ann. de Chimie, Nov. 1841,) shew that

$$\frac{\sigma_0}{\rho_0} = 10463,6,$$

σ_0 being the density of mercury at 0^0, and ρ_0 the density of dry atmospheric air at $0^0 C$, under the pressure of a column of mercury at 0^0, 0,76 metres, or 29,9218 inches high, at the mean level of the sea in latitude 45^0.

At the level of the sea in lat. 45^0 $g = 32.17237$ feet;

$$\therefore \sqrt{\mu} = 916,188 \text{ feet.}$$

33. The expansion of air and of all other permanently gaseous fluids between the temperatures 0^0 and $100^0 C$, under a constant pressure, is equal to 0,3665 of its volume at 0^0 according to the observations of Rudberg and Regnault (Ann. de Chimie, Jan. 1842) and the increment of its volume is nearly proportional to its temperature above 0^0, as indicated by a mercurial thermometer. Therefore, if u_0, u_τ be the spaces occupied by a given mass of air, ρ_0, ρ_τ its densities, at the temperatures 0^0, τ^0, under a constant pressure,

$$u_\tau = \{1 + (0,003665)\,\tau\}\, u_0.$$

Since the quantity of the air is constant,

$$\rho_0 u_0 = \rho_T u_T; \therefore \rho_0 = \{1 + (0{,}003665)\, T\} \rho_T.$$

Hence

$$\Pi = \mu \rho_0 = \mu \{1 + (0{,}003665)\, T\} \rho_T.$$

At the level of the sea in lat. λ

$$g = 32{,}17237 (1 - 0{,}00256 \cos 2\lambda) \text{ feet}$$

$$\sqrt{\mu} = 916{\cdot}118 \text{ feet.}$$

Hence the density of mercury at 0^0 divided by the density of atmospheric air at T^0, under the pressure of a column of mercury at 0^0, h inches high, at the level of the sea in lat. λ

$$= \frac{313089}{h} \cdot \frac{1 + 0{,}003665\, T}{1 - 0{,}00256 \cos 2\lambda}.$$

34. The quantity of a gas is frequently measured by its volume at a given temperature and under a given pressure.

Let u' be the volume of a gas at the temperature T' under the pressure Π'; u its volume at the standard temperature T under the standard pressure Π. The volumes of the gas at the temperature 0^0, under the pressures Π', Π will be

$$\frac{u'}{1 + ET'} \text{ and } \frac{u}{1 + ET}$$

respectively, where $E = 0{,}003665$, the expansion of air for one degree of temperature (33). But the pressure of air at a given temperature varies inversely as the space it occupies (31), therefore

$$\frac{\Pi}{\Pi'} = \frac{u'}{u} \frac{1 + ET}{1 + ET'};$$

$$\therefore \frac{u}{u'} = \frac{\Pi'}{\Pi} \cdot \frac{1 + ET}{1 + ET'}.$$

35. When a given mass of air is compressed or permitted to dilate, it is found that its temperature is increased in the former case and diminished in the latter. Since the density and temperature are both increased or both diminished, it

follows that the alteration of pressure corresponding to a given change of density is greater than if the temperature remained invariable.

Let Π, T, ρ be the pressure, temperature, and density of a given mass of air; Π', T′ its pressure and temperature after its density has been suddenly changed to ρ', where ρ' does not differ much from ρ; $\omega\left(1-\dfrac{\rho}{\rho'}\right)$ the increase of temperature due to the sudden change of density from ρ to ρ'; E the expansion of air for one degree of temperature. Then

$$\begin{aligned}
\Pi &= \mu\rho(1+\mathrm{ET})\\
\Pi' &= \mu\rho'(1+\mathrm{ET}')\\
&= \mu\rho'\{1+\mathrm{ET}+\mathrm{E}(\mathrm{T}'-\mathrm{T})\}\\
&= \mu\rho'\left\{1+\mathrm{ET}+\mathrm{E}\omega\left(1-\frac{\rho}{\rho'}\right)\right\}\\
&= \mu\{(1+\mathrm{ET})\rho'+\mathrm{E}\omega(\rho'-\rho)\};
\end{aligned}$$

$$\therefore\ \frac{\Pi'-\Pi}{\Pi} = \left(1+\frac{\mathrm{E}\omega}{1+\mathrm{ET}}\right)\frac{\rho'-\rho}{\rho}.$$

It appears from experiment that $\mathrm{E}\omega$ is proportional to $1+\mathrm{ET}$, or that

$$1+\frac{\mathrm{E}\omega}{1+\mathrm{ET}}$$

is the same for all values of T. Let this quantity be denoted by K, and let $\rho'=\rho+\delta\rho$. The change of pressure from Π to Π' is due partly to the change of density and partly to the change of temperature which is caused by the change of density. Hence the pressure in passing from Π to Π' is a function of the density only. Therefore $\Pi'=\Pi+d_\rho\Pi\,\delta\rho$ ultimately. Hence

$$\begin{aligned}
\frac{1}{\Pi}d_\rho\Pi &= \mathrm{K}\frac{1}{\rho};\\
\therefore\ \log_e\Pi &= \mathrm{K}\log_e\rho + C,\\
\log_e\Pi' &= \mathrm{K}\log_e\rho' + C;\\
\therefore\ \log_e\frac{\Pi'}{\Pi} &= \mathrm{K}\log_e\frac{\rho'}{\rho};
\end{aligned}$$

$$\therefore \frac{\Pi'}{\Pi} = \left(\frac{\rho'}{\rho}\right)^{\kappa}.$$

$$\frac{\Pi'}{\Pi} = \frac{\rho'}{\rho} \cdot \frac{1 + ET'}{1 + ET};$$

$$\therefore \frac{1 + ET'}{1 + ET} = \left(\frac{\rho'}{\rho}\right)^{\kappa - 1}.$$

It will be seen presently that the probable value of κ is 1,41754.

If, in making the first experiment described in (31), the mercury be suddenly poured into the tube *DB*, the temperature of the air in *AB* (fig. 18) will be increased; and if the altitude *CM* be observed before the air in *AB* has cooled down to its original temperature, the pressure will appear to vary in a higher inverse ratio than that of the first power of the space occupied by the air. A similar observation applies to the second experiment.

36. The specific heats of two different bodies, or of the same body in different states, are proportional to the quantity of heat required to produce a given change of temperature in them, or they are inversely proportional to the change of temperature produced in equal masses of the bodies by equal quantities of heat.

Let T′ be the temperature, ρ' the density of a mass of air under a given pressure Π; let the heat q change the temperature to T, and the density to ρ, the pressure remaining unchanged. Therefore

$$\frac{\rho}{\rho'} = \frac{1 + ET'}{1 + ET}.$$

Let the air be now compressed till it acquires its original density ρ'; the heat evolved will be

$$\omega\left(1 - \frac{\rho}{\rho'}\right) = \frac{E\omega(T - T')}{1 + ET}.$$

Therefore upon the whole the temperature of the air has been increased by the quantity

$$(T - T')\left\{1 + \frac{E\omega}{1 + ET}\right\}.$$

This is the quantity by which the application of the heat q has increased the temperature of the air, its volume remaining constant. Under a constant pressure the heat q increased the temperature of the air by the quantity $\text{T} - \text{T}'$. Hence

$$1 + \frac{\text{E}\omega}{1 + \text{E}\text{T}} = \frac{\text{specific heat of air (press. const.)}}{\text{specific heat of air (vol. const.)}}.$$

37. Let two vessels the capacities of which are u and v be filled with the gases (A) and (B) at the same temperature T and under the same pressure Π, and open a communication between the two vessels: in the course of a short time, provided the gases be not such as act chemically upon each other, it will be found that they are uniformly diffused through both vessels, and that the pressure at the temperature T remains unaltered.

A volume u of a gas (A) under the pressure M is mixed with a volume v of a gas (B) under the pressure Π, to find w, the volume of the mixture, under the pressure P, the temperature remaining unchanged throughout.

Under the pressure P the volumes of (A), (B) will be $\frac{\text{M}}{\text{P}}u$, $\frac{\Pi}{\text{P}}v$, respectively.

But if the gases after being mixed occupy a space equal to the sum of their volumes before they were mixed, their pressure will still be P. Therefore under the pressure P their volume (w) will be

$$\frac{\text{M}}{\text{P}}u + \frac{\Pi}{\text{P}}v; \quad \therefore \; Pw = \text{M}u + \Pi v.$$

$$\text{If } u = v = w, \; \text{P} = \text{M} + \Pi.$$

38. If a small quantity of any liquid capable of affording vapour be introduced into a vessel from which the air has been withdrawn, the vessel will be almost instantly filled with vapour, the pressure and density of which are found to depend only on its temperature, as long as the whole of the liquid is not converted into vapour. If the space in which the vapour

exists be increased, a fresh portion of the liquid will take the form of vapour; and if it be diminished, a portion of the vapour will return to a liquid state; but the pressure and density will remain the same in either case, provided the temperature undergoes no change. If the temperature be increased, a fresh portion of the liquid will be converted into vapour, and the pressure and density will be increased. If the temperature be diminished, a portion of the vapour will return to the state of a liquid, and the density and pressure of the remainder will be diminished. If the space be increased sufficiently, the whole of the liquid will assume the form of vapour. Under these circumstances the relation between the pressure, temperature, and density of vapour, is very nearly the same as for air.

The pressures exerted by the vapours of various fluids in contact with the fluids from which they are produced have been determined experimentally, and empirical formulæ have been constructed, which, for a certain range of temperature, express the results of these experiments with considerable accuracy. Yet hitherto no law has been discovered by which the pressure at any assignable temperature can be determined. Formulæ exhibiting the pressure of the vapour of water in contact with water, at any temperature between certain limits, will be given at the end of the treatise.

It appears probable, from the experiments of Mr Faraday, that every gas may be made to assume the form of a liquid by diminishing its volume. When the condensation of a gas is carried on nearly to the point at which it begins to liquefy, the ratio of its pressure to its density at a given temperature is no longer constant. The value of this ratio for dry atmospheric air does not however perceptibly change under the pressure of a column of mercury nearly ninety feet high. It has also been shewn that the expansibility of gases and vapours is not without limits.

39. When the liquid is introduced into a vessel containing air, precisely the same effects are produced, except that the vapour is formed slowly. The quantity of liquid finally converted into vapour is the same as if the vessel contained no air.

Let M be the pressure of the air before the introduction of the liquid, Υ the pressure which the same quantity of vapour would exert if the vessel contained no air. The volumes of the air, of the vapour, and of the mixture of air and vapour are the same, therefore (37) the pressure of the mixture of air and vapour = M + Υ.

40. If a solid surrounded by a mixture of air and vapour be cooled down below the temperature corresponding to the density of the vapour in the mixture, the stratum of air in immediate contact with the solid will be cooled, and the excess of vapour contained in it will be deposited upon the surface of the solid in the form of dew, which may be made to disappear by heating the solid above the temperature corresponding to the density of the vapour.

The lowest temperature at which the whole of the vapour contained in any mixture of air and vapour is capable of remaining in an elastic state is called the dew-point. It may be determined practically in the following manner: cool a vessel of glass or polished metal, and observe its temperature when dew begins to be deposited upon it; suffer it to grow warm, and observe the temperature at which the dew disappears. These two temperatures are one less and the other greater than the dew-point, and they differ very little from each other; their mean may therefore be considered as the dew-point.

The observation of the dew-point enables us to determine the pressure of the vapour contained in any mixture of air and vapour. For if Υ be the pressure of the vapour, t the temperature, τ the dew-point, Υ_τ the pressure of vapour corresponding to the temperature τ, is known by experiment. At temperatures above τ the relation between the pressure, density, and temperature of the vapour, is the same as for air;

$$\therefore \frac{\Upsilon}{1+et} = \frac{\Upsilon_\tau}{1+e\tau}.$$

The pressure of the vapour of water existing in the atmosphere may be deduced from the dew-point by means of the equation given above. It may also be obtained by observing the temperature of a thermometer the bulb of which is covered with muslin and kept constantly moist. According to August, if t be the temperature of the air, t' the temperature of the wet thermometer, Υ the pressure of the vapour, Υ' the pressure of vapour corresponding to the temperature t',

$$\Upsilon = \Upsilon' - 0{,}02239\,(t - t')\,\frac{\Pi}{28{,}776}.$$

41. To compare the density of a mixture of air and vapour with the density of dry air at the same temperature and under the same pressure.

Let Π be the pressure of the mixture of air and vapour, t its temperature, Υ the pressure of the vapour, and therefore $\Pi - \Upsilon$ the pressure of the air alone, m the ratio of the specific gravity of the vapour to that of the air at the same temperature and pressure, ρ the density of dry air at t^0 under the pressure Π. The density of air at t^0 under the pressure $\Pi - \Upsilon$ will be $\frac{\Pi - \Upsilon}{\Pi}\rho$. The density of vapour at t^0 under the pressure Υ will be $\frac{\Upsilon}{\Pi} m\rho$.

Hence the mass of a volume V of the mixture of air and vapour is equal to a mass $V\frac{\Pi - \Upsilon}{\Pi}\rho$ of dry air together with a mass $V\frac{\Upsilon}{\Pi} m\rho$ of vapour. But the mass of a volume V of dry air at t^0 under the pressure Π is equal to $V\rho$;

$$\therefore\ \frac{\text{density moist air}}{\text{density dry air}} = \frac{1}{\Pi}(\Pi - \Upsilon + m\Upsilon) = 1 - (1 - m)\frac{\Upsilon}{\Pi}.$$

According to Gay Lussac, (density vapour) $\div$ (density air) $= 0{,}625$, therefore $1 - m = 0{,}375$.

Hence, at the same temperature and under the same pressure,

$$\frac{\text{density moist air}}{\text{density dry air}} = 1 - 0{,}375\,\frac{\Upsilon}{\Pi},$$

where Π is the pressure of the moist air, and Υ the pressure of the vapour it contains.

42. Having given the volume of a mixture of air and vapour, to find the volume which the air alone would occupy under the same pressure and at the same temperature.

Let u be the volume of the mixture, u' the volume of the air alone under the same pressure, Π its pressure, Υ the pressure of the vapour; therefore $\Pi - \Upsilon$ is the pressure which the air would exert if the vapour were removed. But the space occupied by air is inversely proportional to the pressure, therefore, under the pressure Π the volume of the air alone

$$= \frac{\Pi - \Upsilon}{\Pi} u.$$

43. It appears from reasoning similar to that employed in (8) that when an elastic fluid of uniform temperature, acted on by gravity, is at rest, its pressure, and therefore its density, is the same at all points in the same horizontal plane. This is also true when the temperature at any point depends only on the distance of the point from a given horizontal plane.

44. To find the difference of the altitudes of two stations by means of the barometer.

Let H, Π, K be the pressures of the air at the points H, P, K in the vertical HPK (fig. 20); S the temperature at H, T the temperature at K; Q a point very near to P; $HK = z$, $MP = x$, $PQ = \delta x$; then $\Pi + d_x \Pi \delta x$ will ultimately be the pressure at Q, g the force of gravity, E the expansion of air for one degree of heat under a constant pressure. The temperature of the air decreases slowly as we ascend; it may however be supposed uniform between H and K, and equal to the mean temperature $\frac{1}{2}$ (S + T) without causing any considerable error. The density of the air at P

$$= \frac{\Pi}{\mu} \frac{1}{1 + \frac{1}{2} \mathrm{E} (\mathrm{S} + \mathrm{T})}.$$

But pressure at $P = g$ (density at P) PQ + pressure at Q (9), or

$$\Pi = g \frac{\Pi}{\mu} \frac{\delta x}{1 + \frac{1}{2} \mathrm{E} (\mathrm{S} + \mathrm{T})} + \Pi + d_x \Pi \delta x;$$

$$\therefore \frac{1}{\Pi} d_x \Pi = -\frac{g}{\mu} \frac{1}{1 + \frac{1}{2}E(S + T)};$$

$$\therefore \log_e \Pi = C - \frac{g}{\mu} \frac{x}{1 + \frac{1}{2}E(S + T)}.$$

$\log_e K = C - \frac{g}{\mu} \frac{z}{1 + \frac{1}{2}E(S + T)}$, $\log_e H = C$, for $\Pi = K$ when $x = z$, and $\Pi = H$ when $x = 0$,

$$\therefore \log \frac{H}{K} = \frac{g}{\mu} \frac{z}{1 + \frac{1}{2}E(S + T)}.$$

Let h be the altitude, s the temperature of the mercury in the barometer at H; k the altitude, t the temperature of the mercury at any point in a horizontal plane passing through K; e the expansion of mercury for one degree of temperature; then

$$\frac{H}{K} = \frac{h}{k} \frac{1 + et}{1 + es} = \frac{h}{k} \{1 - e(s - t)\}.$$

$$\log_e \frac{H}{K} = \log_e 10 . \log_{10} \frac{H}{K}$$

$$= \log_e 10 . [\log_{10} h - \log_{10} k + \log_{10} \{1 - e(s - t)\}]$$

$$= \log_e 10 \left\{\log_{10} h - \log_{10} k - \frac{e(s - t)}{\log_e 10}\right\} \text{ nearly.}$$

45. If λ be the latitude of the place of observation, and f the force of gravity in lat. 45^0,

$$g = f(1 - 0{,}00256 \cos 2\lambda).$$

Hence

$$z = \log_e 10 \frac{\mu}{f} \frac{1 + \frac{1}{2}E(S + T)}{1 - 0{,}00256 \cos 2\lambda} \left\{\log_{10} h - \log_{10} k - \frac{e(s - t)}{\log_e 10}\right\}.$$

By comparing a number of corresponding values of z, h, k,... it has been found that

$$\log_e 10 . \frac{\mu}{f} = 60345 \text{ feet}, \quad 60345 . 0{,}00256 = 155,$$

$$60345 \tfrac{1}{2}E = 120, \quad \frac{e}{\log_e 10} = 0{,}000078,$$

the temperatures being expressed in degrees of the centigrade thermometer.

Hence, if x be the number of feet contained in HK, expanding and neglecting products and powers of the small terms after the first,

$$x = \left\{60345 + 120\,(\text{S} + \text{T}) + 155 \cos 2\lambda\right\}$$
$$\times \left\{\log_{10} h - \log_{10} k - 0{,}000078\,(s - t)\right\}.$$

46. When the difference of the altitudes of the stations is large, it becomes necessary to take into account the variation of gravity in the same vertical. Let r be the distance of H from the center of the earth; g' the force of gravity at P, g the force of gravity at H. It has been found that when P is on the surface of the ground, and we take into account the attraction of the portion of the earth which is elevated above the level of H,

$$g' = g\left(1 - \tfrac{5}{4}\frac{x}{r}\right) \text{ nearly};$$

$$\therefore \frac{1}{\Pi} d_x \Pi = -\frac{g'}{\mu} \frac{1}{1 + \frac{1}{2}\text{E}\,(\text{S} + \text{T})} = -\frac{g}{\mu} \frac{1 - \frac{5}{4}\frac{x}{r}}{1 + \frac{1}{2}\text{E}\,(\text{S} + \text{T})};$$

$$\therefore \log_e \Pi = C - \frac{g}{\mu} \frac{x - \frac{5}{8}\frac{x^2}{r}}{1 + \frac{1}{2}\text{E}\,(\text{S} + \text{T})}.$$

$$\log_e \text{H} = C, \quad \log_e \text{K} = C - \frac{g}{\mu} \frac{x - \frac{5}{8}\frac{x^2}{\mu}}{1 + \frac{1}{2}\text{E}\,(\text{S} + \text{T})};$$

$$\therefore \log_e \frac{\text{H}}{\text{K}} = \frac{g}{\mu} \frac{x\left(1 - \frac{5}{8}\frac{x}{r}\right)}{1 + \frac{1}{2}\text{E}\,(\text{S} + \text{T})} = \frac{g}{\mu} \frac{x}{1 + \frac{1}{2}\text{E}\,(\text{S} + \text{T})} \frac{1}{1 + \frac{5}{8}\frac{x}{r}} \text{ nearly}.$$

$$\text{But } \frac{\text{H}}{\text{K}} = \frac{g}{g'} \frac{h}{k} \frac{1 + et}{1 + es} = \frac{1}{1 - \frac{5}{4}\frac{x}{r}} \frac{h}{k} \frac{1 + et}{1 + es} = \frac{h}{k}\left\{1 - e\,(s - t) + \tfrac{5}{4}\frac{x}{r}\right\};$$

$$\therefore \log_e \frac{\text{H}}{\text{K}} = \log_e 10 \left[\log_{10} h - \log_{10} k - \frac{1}{\log_e 10}\left\{e\,(s - t) - \tfrac{5}{4}\frac{x}{r}\right\}\right];$$

$$\therefore\ z = \log_e 10 \frac{\mu}{f} \frac{1 + \frac{1}{2}\text{E}(\text{S} + \text{T})}{1 - 0{,}00256 \cos 2\lambda} \left(1 + \tfrac{5}{8}\frac{z}{r}\right)$$

$$\times \left[\log_{10} h - \log_{10} k - \frac{1}{\log_e 10}\left\{e(s-t) - \tfrac{5}{4}\frac{z}{r}\right\}\right].$$

47. If

$$z' = 64000\ \{\log_{10} h - \log_{10} k - 0{,}000078\,(s-t)\},$$

z' may be substituted for z in the terms having small coefficients;

$$\frac{1}{\log_e 10}\,\tfrac{5}{4}\,\frac{64000}{r} = 0{,}00166,$$

$$\therefore\ \log_{10} h - \log_{10} k - 0{,}000078\,(s-t) + \frac{1}{\log_e 10}\,\tfrac{5}{4}\,\frac{z'}{r}$$

$$= 1{,}00166\ \{\log_{10} h - \log_{10} k - 0{,}000078\,(s-t)\}.$$

By substituting observed values of z, h, k in the equation between z, h, k, it has been found that

$$1{,}00166 \log_e 10 \frac{\mu}{f} = 60258,\quad 60258 \tfrac{1}{2}\text{E} = 120,\quad 60258 . \tfrac{5}{8}\,\frac{64000}{r} = 115.$$

Whence

$$z = \{60258 + 120\,(\text{S} + \text{T}) + 155 \cos 2\lambda + 115\,(\log_e h - \log_e k)\}$$

$$\times\ \{\log_{10} h - \log_{10} k - 0{,}000078\,(s-t)\}.$$

The constants in (45) (47) are adapted to the mixture of air and watery vapour constituting the atmosphere in its ordinary state. The vapour of water is lighter than dry air, and the quantity of vapour contained in a given quantity of atmospheric air increases with the temperature. Hence μ and E are larger than if the atmosphere consisted of perfectly dry air.

48. When the pressure of the vapour contained in the atmosphere at the upper and lower stations is known from observations of the dew-point, or of the temperature of a thermometer with a wet bulb, the difference of the altitudes of the stations may be found with greater accuracy.

Let w be the mean of the pressures of the vapour divided by the mean of the atmospheric pressures at the upper and lower stations. Then, at the same temperature and under the same pressure,

$$\frac{\text{density atmospheric air}}{\text{density dry air}} = 1 - 0{,}375w \text{ nearly (41.)};$$

$$\therefore \text{ density atmospheric air at } P = \frac{\Pi}{\mu} \cdot \frac{1 - 0{,}375w}{1 + \frac{1}{2}\mathrm{E}(\mathrm{S} + \mathrm{T})}.$$

Proceeding as in (44.) with this value of the density, we obtain

$$x = \log_e 10 \cdot \frac{\mu}{g} \cdot \frac{1 + \frac{1}{2}\mathrm{E}(\mathrm{S}+\mathrm{T})}{1 - 0{,}375w}\left(1 + \tfrac{5}{8} \cdot \frac{x}{r}\right)$$

$$\times \left[\log_{10} h - \log_{10} k - \frac{1}{\log_e 10}\left\{e(s - t) - \tfrac{5}{4} \cdot \frac{x}{r}\right\}\right].$$

$$\log_e 10 = 2{,}3025851, \quad \mathrm{E} = 0{,}003665.$$

$$\left.\begin{array}{l} \sqrt{(\mu)} = 916{,}188 \\ g = 32{,}17237\,(1 - 0{,}00256 \cos 2\lambda) \\ r = 20888761 \end{array}\right\} \text{feet.}$$

Whence $x =$

$$\{60076 + 22529w + 110(\mathrm{S}+\mathrm{T}) + 155\cos 2\lambda + 115(\log_{10} h - \log_{10} k)\}$$
$$\times \{\log_{10} h - \log_{10} k - 0{,}000078\,(s - t)\}.$$

SECTION IV.

ON THE EQUILIBRIUM OF FLUIDS ACTED ON BY ANY FORCES.

ART. 49. To find the pressure at any point in a mass of fluid at rest acted on by any forces.

Let PQ (fig. 21.) be the edge of a very small prism of fluid in the interior of a mass of fluid at rest, R the accelerating force at P, S the resolved part of R in the direction PQ. Let the prism become solid; then, since S.(mass prism), and the pressures on its ends are the only forces that act upon it in a direction parallel to PQ, they must be in equilibrium;

$\therefore$ press. on the end Q − press. on the end $P = S$.(mass prism).

Let x, y, z; $x + \delta x$, $y + \delta y$, $z + \delta z$ be the co-ordinates of P, Q referred to rectangular axes Ox, Oy, Oz. Construct a parallelopiped LMN, of which PQ is the diagonal, having its edges PL, PM, PN parallel to Ox, Oy, Oz respectively. Let X, Y, Z be the components of R resolved parallel to Ox, Oy, Oz; κ the area of the base of the prism; ρ the density of the fluid; p the pressure at P, and therefore, ultimately, $p + d_x p \,.\, \delta x + d_y p \,.\, \delta y + d_z p \,.\, \delta z$, the pressure at Q. If the sides of the base of the prism be very small compared with its length, pressure on the end Q − pressure on the end P

$$= \kappa (d_x p \,.\, \delta x + d_y p \,.\, \delta y + d_z p \,.\, \delta z).$$

But $S = X \,.\, \cos QPL + Y \,.\, \cos QPM + Z \,.\, \cos QPN$;

and the mass of the prism $= \rho\kappa \,.\, PQ$;

$$\therefore\ S \,.\, (\text{mass prism})$$

$$= \rho\kappa \,.\, PQ \,.\, (X \cos QPL + Y \cos QPM + Z \cos QPN)$$

$$= \rho\kappa (X \,.\, \delta x + Y \,.\, \delta y + Z \,.\, \delta z);$$

$$\therefore\ d_x p \,.\, \delta x + d_y p \,.\, \delta y + d_z p \,.\, \delta z = \rho (X \,.\, \delta x + Y \,.\, \delta y + Z \,.\, \delta z).$$

δx, δy, δz are independent of each other; therefore

$$d_x p = \rho X, \quad d_y p = \rho Y, \quad d_z p = \rho Z;$$

if, then, we can find a quantity p, such that

$$d_x p = \rho X, \quad d_y p = \rho Y, \quad d_z p = \rho Z,$$

p, taken between the proper limits, is the pressure at P.

50. When the fluid is elastic $p = \mu\rho$;

$$\therefore \frac{\mu}{p} . d_x p = X, \quad \frac{\mu}{p} d_y p = Y, \quad \frac{\mu}{p} d_z p = Z,$$

therefore p is a quantity such that

$$\mu d_x \log_e p = X, \quad \mu d_y \log_e p = Y, \quad \mu d_z \log_e p = Z,$$

and if u be a quantity such that

$$d_x u = X, \quad d_y u = Y, \quad d_z u = Z,$$

$$p = Ce^{\frac{u}{\mu}}.$$

51. $$d_x p = \rho X, \quad d_y p = \rho Y, \quad d_z p = \rho Z;$$

$$d_z d_y p = d_y d_z p, \quad d_x d_z p = d_z d_x p, \quad d_y d_x p = d_x d_y p;$$

$$\therefore d_z(\rho Y) = d_y(\rho Z),$$

$$d_x(\rho Z) = d_z(\rho X),$$

$$d_y(\rho X) = d_x(\rho Y).$$

If we perform the differentiations, and multiply the first equation by X, the second by Y, and the third by Z, and add, we obtain

$$X(d_z Y - d_y Z) + Y(d_x Z - d_z X) + Z(d_y X - d_x Y) = 0.$$

This equation expresses the relation that must exist between the forces X, Y, Z, in order that the equilibrium may be possible.

When the density is constant,

$$d_y X = d_x Y, \quad d_z X = d_x Z, \quad d_z Y = d_y Z.$$

52. If c be the pressure at any point P in the fluid, $p = c$, in which z is an implicit function of x and y, is the equation to the surface of equal pressure passing through P.

The derived equations of $p = c$ are

$$d_{(x)}p + d_{(z)}p \,.\, d_x z = 0, \quad d_{(y)}p + d_{(z)}p \,.\, d_y z = 0,$$

x, y, z being considered independent of each other in forming the differential coefficients $d_{(x)}p$, $d_{(y)}p$, $d_{(z)}p$;

$$\therefore\ X + Z \,.\, d_x z = 0, \quad Y + Z \,.\, d_y z = 0;$$

therefore if α, β, γ be the angles between Ox, Oy, Oz, and the normal at P, to the surface of equal pressure passing through P,

$$\cos\alpha = \frac{X}{R}, \quad \cos\beta = \frac{Y}{R}, \quad \cos\gamma = \frac{Z}{R},$$

where $R^2 = X^2 + Y^2 + Z^2$. But $\frac{X}{R}$, $\frac{Y}{R}$, $\frac{Z}{R}$, are the cosines of the angles between Ox, Oy, Oz, and the direction in which the force at P acts. Therefore the force at any point acts in the direction of a normal to the surface of equal pressure at that point.

The equation to the surface of a fluid is $p = 0$. And if the fluid be contiguous to another fluid with which it does not mix, and which exerts a pressure Π at the common surface of the fluids, $p = \Pi$ will be the equation to the common surface of the fluids.

53. When ρ is variable, and a quantity u can be found, such that $X = d_x u$, $Y = d_y u$, $Z = d_z u$, ρ must be a function of u. For

$$d_x p = \rho d_x u, \quad d_y p = \rho d_y u, \quad d_z p = \rho d_z u,$$

and these equations cannot be satisfied unless ρ is a function of u.

Let $\rho = fu$, then $d_x p = fu d_x u$; $\therefore\ p = \int_u fu$. Hence u and ρ are functions of p; and when p is constant, u and ρ must be constant; therefore $u = c$ is the equation to a surface of equal pressure; also ρ is the same at all points in a surface of equal pressure.

Hence when an elastic fluid of variable temperature is at rest, the temperature is the same at all points in a surface of equal pressure.

54. The conditions $d_y X = d_x Y$, $d_z X = d_x Z$, $d_y Z = d_z Y$, are satisfied whenever the forces tend to fixed centers, and the intensity of each force at any point P, is a function of the distance of P from the center to which the force tends.

For if a, b, c be the co-ordinates of the center to which one of the forces tends, r its distance from P, ϕr the intensity of the force at P,

$$X = \Sigma\left(\phi r \frac{x-a}{r}\right), \quad Y = \Sigma\left(\phi r \frac{y-b}{r}\right), \quad Z = \Sigma\left(\phi r \frac{z-c}{r}\right),$$

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2;$$

$$\therefore\ d_y X = \Sigma\left\{\left(d_r \phi r - \frac{1}{r}\phi r\right)\frac{x-a}{r}.\frac{y-b}{r}\right\}.$$

We should have obtained the same expression for $d_x Y$, therefore $d_y X = d_x Y$; in like manner $d_z X = d_x Z$, and $d_y Z = d_z Y$.

55. $u = \Sigma(\int_r \phi r)$, retaining the notation of (54). For if $u = \Sigma(\int_r \phi r)$,

$$d_x u = \Sigma\left(\phi r.\frac{x-a}{r}\right) = X,$$

$$d_y u = \Sigma\left(\phi r.\frac{y-b}{r}\right) = Y,$$

$$d_z u = \Sigma\left(\phi r.\frac{z-c}{r}\right) = Z.$$

56. Each particle of a fluid attracts with a force which vanishes when the distance of the particle from the attracted point is finite; to find the pressure at any point in the interior of the fluid.

Let the plane xOy (fig. 22) be a tangent to the surface of the fluid touching it in O, Oz perpendicular to xOy; xOz, yOz, the planes of greatest and least curvature. Through any point M in the surface of the fluid, draw NMQ parallel to Oz, meeting xOy in N.

Let R, S be the radii of curvature of the surface of the fluid in the planes xOz, yOz; $NOx = \theta$, $NO = \rho$, $NM = z$, $OP = r$, $MP = u$, $NQ = z_{,}$, $PQ = u_{,}$; ϕu the attraction of a portion of the fluid the volume of which is unity on a point at the distance u, V the attraction of the fluid on P, which manifestly acts in the direction Oz.

The equation to the surface of the fluid is

$$z = \tfrac{1}{2}\rho^2\left\{\frac{(\cos\theta)^2}{\text{R}} + \frac{(\sin\theta)^2}{\text{S}}\right\},$$

$$\text{and } u_{,}^2 = \rho^2 + (z_{,} - r)^2, \quad u^2 = \rho^2 + (z - r)^2;$$

$$\therefore\ u_{,}.d_{z_{,}}u_{,} = z_{,} - r, \text{ and } u^2 = \rho^2 + r^2 - 2rz \text{ nearly};$$

$$\therefore\ ud_{\rho}u = \rho - rd_{\rho}z = \rho\left\{1 - r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\};$$

$$\therefore\ \rho = ud_{\rho}u\left\{1 + r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\} \text{ nearly}.$$

$$V = \int_{\theta}^{0\ 2\pi}\int_{\rho}^{0}\int_{z_{,}}^{z\ \infty}\rho\frac{z_{,} - r}{u_{,}}\phi u_{,},$$

$$\int_{z_{,}}\frac{z_{,} - r}{u_{,}}\phi u_{,} = \int_{z_{,}}\phi u_{,}d_{z_{,}}u_{,} = \int_{u_{,}}\phi u_{,};$$

when $z_{,} = \infty$, $u_{,} = \infty$; when $z_{,} = z$, $u_{,} = u$;

$$\therefore\ \int_{z_{,}}^{z\ \infty}\frac{z_{,} - r}{u_{,}}\phi u_{,} = \int_{u_{,}}^{u\ \infty}\phi u_{,} = \varpi u.$$

$$\int_{\rho}\rho\varpi u = \int_{\rho}ud_{\rho}u\left\{1 + r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\}\varpi u,$$

$$= \left\{1 + r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\}\int_{u}u.\varpi u;$$

when $\rho = \infty$, $u = \infty$; when $\rho = 0$, $u = r$;

$$\therefore\ {}^{0}\!\int_{p}^{\infty} \rho\varpi u = \left\{1 + r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\} {}^{r}\!\int_{u}^{\infty} u.\varpi u$$

$$= \left\{1 + r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\}\psi r.$$

$${}^{0}\!\int_{0}^{2\pi} \left\{1 + r\left(\frac{\overline{\cos\theta}|^2}{\text{R}} + \frac{\overline{\sin\theta}|^2}{\text{S}}\right)\right\} = 2\pi\left\{1 + \frac{r}{2}\left(\frac{1}{\text{R}} + \frac{1}{\text{S}}\right)\right\};$$

$$\therefore\ V = 2\pi\left\{1 + \frac{r}{2}\left(\frac{1}{\text{R}} + \frac{1}{\text{S}}\right)\right\}\psi r.$$

Since the attraction of the fluid is insensible at sensible distances, ϖu decreases with extreme rapidity as u increases, and vanishes when the value of u becomes sensible. The same remark applies to ψr.

If the density of the fluid $= D$, the pressure at P arising from the attraction of the fluid

$$= D\,{}^{0}\!\int_{r}^{r} V = D\left\{2\pi\,{}^{0}\!\int_{r}^{r} \psi r + \pi\left(\frac{1}{\text{R}} + \frac{1}{\text{S}}\right)\right\}{}^{0}\!\int_{r}^{r} r\psi r.$$

$$\text{Let } 2\pi\,{}^{0}\!\int_{r}^{\infty} \psi r = K, \quad 2\pi\,{}^{0}\!\int_{r}^{\infty} r\psi r = H;$$

then, since the force becomes insensible at sensible distances from the surface of the fluid, the pressure at P, arising from the attraction of the fluid, remains constant for all sensible values of OP; $\therefore\ 2\pi\,{}^{0}\!\int_{r}^{r} \psi r$ and $2\pi\,{}^{0}\!\int_{r}^{r} r\psi r$ become K and H, as soon as r becomes sensible; therefore when OP is finite, the pressure at P, produced by the attraction of the fluid,

$$= D\left\{K + \tfrac{1}{2}H\left(\frac{1}{\text{R}} + \frac{1}{\text{S}}\right)\right\}.$$

When the surface of the fluid is a plane, $\frac{1}{\text{R}} = 0$, $\frac{1}{\text{S}} = 0$, and the pressure $= D.K$.

When the surface of the fluid is concave, R and S become negative.

Since ψr vanishes when r becomes sensible, $r\psi r$ is much less than ψr, therefore $\frac{1}{2}H\left(\frac{1}{R}+\frac{1}{S}\right)$ is much less than K, or, the attraction of a fluid on a particle of fluid in its surface, is nearly independent of the curvature of its surface.

57. Let ACD (fig. 23) be a narrow cylindrical tube, partly filled with a fluid acted on by no forces except its own attraction, and the attraction of the tube. Let mT be the attraction of the matter of which the tube is made on a particle of fluid in its surface, nT the attraction of the fluid on a particle of fluid in its surface, the surface in which the particle is placed being either a plane or a surface of continuous curvature.

(1) Let the surface of the fluid in the tube be a plane ABC perpendicular to the axis of the tube. Draw AD parallel to the axis of the tube, AC a diameter of the circle ABC, and AG bisecting the angle CAD. The attraction of the tube on a particle of the fluid at A is equal to mT, and it acts in the direction CA. Let $\tau\delta\theta$ be the attraction of a wedge of fluid having a very small angle $\delta\theta$ on a particle of fluid in its edge; t the attraction of a wedge of fluid having an angle 2θ on a particle in its edge. It is easily seen that the attraction of a wedge having an angle $2\theta + 2\delta\theta$ will be $t + 2\tau\delta\theta.\cos\theta$. Hence $d_\theta t = 2\tau\cos\theta$; $\therefore\ t = 2\tau\sin\theta$. By making $\theta = \frac{1}{2}\pi$ we obtain the attraction of the fluid on a particle of fluid in its surface. But this is nT, therefore $2\tau = nT.CAD = \frac{1}{2}\pi$; therefore the attraction of the fluid $CBAD$ on a particle of the fluid at A is equal to $nT.\sin\frac{1}{4}\pi$; and the resolved parts of this attraction in the directions AD, AC are each equal to $nT(\sin\frac{1}{4}\pi)^2$, or $\frac{1}{2}nT$. But the whole attraction on a particle of the fluid at A, must be perpendicular to the surface of the fluid at A, or in the direction AD, therefore we must have $\frac{1}{2}nT = mT$, or $n = 2m$.

(2) Let the surface of the fluid in the tube be a concave hemisphere AEC. Complete the sphere $AECF$. The attraction on a particle of the fluid at A will not be sensibly altered if we suppose the upper part of the tube to be filled with fluid, leaving the spherical space $AFCE$ vacant. But

in order that the fluid surrounding *AFCE* may be in equilibrium, the attraction on each particle in its surface must be the same, and perpendicular to the surface, therefore the attraction of the cylinder on a particle of the fluid at *A* must be equal to the attraction of the fluid on a particle of the fluid in its surface, or $nT = mT$, $\therefore n = m$.

When n is greater than m, it is probable that a layer of fluid adheres to the inner surface of the solid tube. The attraction of this fluid tube on a point in its surface is nT, and consequently the surface of the fluid contained in it is a concave hemisphere.

(3) Let the surface of the fluid in the tube be a convex hemisphere *AFC*. Complete the sphere *AFCE*. The attraction on a particle of the fluid at *A* will not be sensibly altered if we remove the fluid in *AECD*, leaving the sphere *AFCE*. But in order that the fluid sphere *AFCE* may be in equilibrium, the attraction on a particle at *A* must be equal to the attraction on a particle at any point *F* in its surface. Therefore the attraction of the tube on a particle of the fluid at *A* must vanish, or $m = 0$.

The surface of water, alcohol, &c. contained in a glass tube of very small diameter is found to be a concave hemisphere. The surface of mercury in such a tube is a convex hemisphere. The surface of mercury which has undergone a change in consequence of having been boiled for a long time in contact with atmospheric air, is a plane perpendicular to the axis of the tube. Tubes such as those mentioned above are called Capillary Tubes.

58. When the lower extremity of a capillary tube is immersed in fluid, the surface of the fluid within the tube is elevated above, or depressed below, the surface of the surrounding fluid, according as it is concave or convex. Thus water is elevated, and mercury depressed in glass tubes. The attraction on which this phenomenon depends, is insensible at sensible distances: for the elevation or depression of the fluid is independent of the thickness of the tube; and the ascent of water in glass tubes is entirely prevented by a thin film of oil.

59. To determine the surface of a fluid contained in a vertical capillary tube.

Let AC (fig. 24) be the axis of the tube meeting the plane of the surface of the exterior fluid in C; APB a section of the surface of the fluid, which will manifestly be a surface of revolution, and $BDB'D'$ a section of the tube made by a plane through AC; PN parallel to AC, AN perpendicular to AC; $DC = a$, $AC = c$, $AN = x$, $PN = y$; b the radius of curvature at A; R, S the radii of curvature at P in the plane APD, and perpendicular to APD; PEA a canal leading from P to A. Now when the fluid in PEA is at rest, the pressure at A produced by the action of gravity and the attraction at P on the fluid in PEA, must be equal to the pressure at A produced by the attraction at A on the fluid in AEP;

$$\therefore\ K - \tfrac{1}{2}H\left(\frac{1}{\mathrm{R}} + \frac{1}{\mathrm{S}}\right) + g\,.\,PN = K - \tfrac{1}{2}H\frac{2}{b};$$

$$\therefore\ 2\frac{g}{H}yx + \frac{2x}{b} = \frac{x d_x^2 y}{\{1 + (d_x y)^2\}^{\frac{3}{2}}} + \frac{d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}};$$

$$\therefore\ 2\frac{g}{H}.\int_x yx + \frac{x^2}{b} = \frac{x d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}} \qquad (1).$$

In order that the fluid in a canal leading from A to a point in the surface of the exterior fluid, may be in equilibrium, we must have $K - \tfrac{1}{2}H\frac{2}{b} + g.AC = K$; $\therefore\ H = gbc$;

$$\therefore\ \frac{g}{H}(cx^2 + 2\int_x yx) = \frac{x d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}}.$$

If α be the angle between a tangent to APB at B and AN, V the volume of the fluid in the tube elevated above the surface of the exterior fluid,

$$\frac{g}{H}\{ca^2 + 2\int_x^{0,a} yx\} = a\sin\alpha;\ \text{and}\ V = \pi c a^2 + 2\pi\int_x^{0,a} yx;$$

$$\therefore\ V = \frac{H}{g}\pi a \sin\alpha.$$

α depends only on the nature of the fluid, and of the substance of which the tube is formed. When the fluid is water, and the tube of glass, $V = (0{,}023444)\pi a$, a being expressed in linear inches, and V in cubic inches. Also when a is small compared with c, the surface of the fluid is a concave hemisphere, $\therefore V = \pi a^2 c + \frac{1}{3}\pi a^3$, $\therefore ac + \frac{1}{3}a^2 = 0{,}023444$.

If $d_x y = \tan\theta$, and a be very small compared with c, $\int_x yx$ is small compared with cx^2, and

$$cx = \frac{H}{g}\sin\theta\,(1 + \frac{2}{cx^2}\int_x yx)^{-1} = \frac{H}{g}\sin\theta\,(1 - \frac{2}{cx^2}\int_x yx)$$

very nearly.

$$cx = \frac{H}{g}\sin\theta,\ d_\theta y = \frac{H}{cg}\sin\theta,\ y = \frac{H}{cg}(1 - \cos\theta) \text{ nearly.}$$

$$\int_x^{0\ \ a} yx = \int_\theta^{0\ \ a} yx d_\theta x = \left(\frac{H}{cg}\right)^3 \{\tfrac{1}{2}(\sin a)^2 - \tfrac{1}{3} + \tfrac{1}{3}(\cos a)^3\};$$

$$\therefore ca = \frac{H}{g}\sin a\,\{1 - \frac{2}{ca^2}\cdot\left(\frac{H}{cg}\right)^3 [\tfrac{1}{2}[\sin a)^2 - \tfrac{1}{3} + \tfrac{1}{3}(\cos a)^3]\};$$

$$\therefore c = \frac{H}{g}\frac{\sin a}{a}\left\{1 - \frac{a}{c\sin a}\left[1 - \frac{2}{3}\frac{1 - (\cos a)^3}{(\sin a)^2}\right]\right\} \quad (3) \text{ nearly.}$$

60. To determine the surface of a fluid between two parallel vertical plates.

Let $D'APD$ (fig. 24) be a section of the surface of the fluid and of the parallel plates, made by a plane perpendicular to their surfaces; AC equidistant from the parallel plates, meeting the surface of the interior fluid in A, and the plane of the surface of the exterior fluid in C. Then, the rest of the construction and the notation being the same as in (59),

$$2\frac{g}{H}y + \frac{1}{b} = \frac{d_x^2 y}{\{1 + (d_x y)^2\}^{\frac{3}{2}}};$$

$$\therefore 2\frac{g}{H}\int_x y + \frac{x}{b} = \frac{d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}} \quad (4);$$

$$\text{and } H = 2gbc;\ \therefore 2\frac{g}{H}(cx + \int_x y) = \frac{d_x y}{\{1 + (d_x y)^2\}^{\frac{1}{2}}};$$

$$\therefore 2\frac{g}{H}\{ca + \int_x^{0\ \ a} y\} = \sin a, \text{ or area } D'B'ADB = \frac{H}{g}\sin a \quad (5).$$

When the fluid is water, and the plates of glass, and very close to each other, $B'AB$ is a semicircle;

$\therefore$ area $DB'ABD = 2ac + 2a^2 - \frac{1}{2}\pi a^2 = 2ac + \frac{3}{7}a^2$ nearly,

$$\text{and } \frac{H}{g} = 0{,}023444, \quad \therefore\ 2ac + \tfrac{3}{7}a^2 = 0{,}023444.$$

If $d_x y = \tan\theta$, then, when a is small,

$$cx = \frac{H}{2g}\sin\theta\left(1 - \frac{1}{cx}\int_x y\right) \text{ very nearly; and}$$

$$\int_x^{a} y = \int_\theta^{a} y\, d_\theta x = \left(\frac{H}{2gc}\right)^2 \left(\sin a - \frac{a}{2} - \tfrac{1}{4}\sin 2a\right);$$

$$\therefore\ ca = \frac{H}{2g}\sin a\left\{1 - \frac{1}{ca}\left(\frac{H}{2gc}\right)^2\left(\sin a - \frac{a}{2} - \tfrac{1}{4}\sin 2a\right)\right\};$$

$$\therefore\ c = \frac{H}{g}\frac{\sin a}{2a}\left\{1 - \frac{a}{c(\sin a)^2}\left(\sin a - \frac{a}{2} - \tfrac{1}{4}\sin 2a\right)\right\} \quad (6).$$

It appears that the elevation of a fluid between two parallel plates, is nearly half the elevation in a tube the diameter of which is equal to the distance between the plates.

When a single plate is immersed vertically in a fluid

$$2\frac{g}{H}y = \frac{d_x^2 y}{\{1 + (d_x y)^2\}^{\frac{3}{2}}}.$$

This is the differential equation to the elastic curve.

The investigation of the form of the surface of the fluid when it is depressed, leads to precisely the same equations as when it is elevated, the sign of y being changed.

If V be the space between the surface of the mercury in a vertical glass tube and the plane of the surface of the mercury on the outside, and a the radius of the tube, $V = (0{,}01)\,\pi a$.

61. To find the tension of a flexible cylindrical vessel containing fluid.

Let MK, PQ, HL (fig. 25) be equidistant sections of the cylinder made by planes perpendicular to its axis. Draw PE, QE normals at the extremities of the small arc PQ; MPH, KQL perpendicular to PEQ; and let p be the pressure of the fluid at P, $t.MH$ the tension of MH or KL, r the radius of curvature of PQ. Now ML is kept at rest by the pressure of the fluid, and the tensions of its edges; the tensions of MH, *KL*, and the pressure of the fluid, are the only forces that act in the plane *PEQ*; the tensions act perpendicular to *PE*, *QE* respectively, and the pressure of the fluid acts perpendicular to *PQ*; therefore, ultimately,

$$\frac{EP}{PQ} = \frac{t.MH}{p.(\text{area } ML)} = \frac{t}{p.PQ}; \quad \therefore\ t = pr.$$

62. To find the tension of a vessel of any form containing fluid.

Let PCP', QCQ' (fig. 26) be the normal sections of least and greatest curvature of the vessel at C; $P'C = PC$, $Q'C = QC$; PE, $P'E$, QF, $Q'F$ normals at the extremities of the small arcs PCP', QCQ'; $MP'K$, HPL sections of the vessel made by planes perpendicular to PEP'; $MQ'H$, KQL sections of the vessel made by planes perpendicular to QFQ'. Let p be the pressure of the fluid at C, $t.QQ'$ the tension of HL or MK, $v.PP'$ the tension of MH or KL; r, s the radii of curvature of PCP', QCQ'. ML is kept at rest by the pressure of the fluid, and the tensions of its edges; therefore the resultant of the tensions must be equal and opposite to the pressure of the fluid. The resultant of the tensions of MK, HL

$$= 2t.QQ'.\sin PEC = \frac{t}{r}.PP'.QQ'.$$

The resultant of the tensions of MH, KL

$$= 2v.PP'.\sin QFC = \frac{v}{s}.PP'.QQ'.$$

The resultants of the tensions act in the direction CE; the pressure of the fluid $= p.PP'.QQ'$, and it acts in the direction EC;

$$\therefore\ p = \frac{t}{r} + \frac{v}{s}.$$

When the tensions are the same in every direction, or $v = t$,

$$p = t\left(\frac{1}{r} + \frac{1}{s}\right).$$

When the vessel is immersed in fluid, p is the difference of the pressures of the interior and exterior fluids.

SECTION V.

ON THE MOTION OF FLUIDS.

ART. 63. WHEN an incompressible fluid flows through a tube the velocities of the fluid at any two points, are inversely proportional to the areas of the perpendicular sections of the tube at those points; supposing the tube to continue always full, and the velocities at all points in the same section to be equal to one another, and perpendicular to the section.

For equal volumes of fluid must pass through each section in the same time; and if u, v be the velocities at the two sections; H, K the areas of the sections; $u\mathrm{H}t$, $v\mathrm{K}t$ will be the volumes of the fluid that passes through the two sections in the small time t; and these are equal; $\therefore u\mathrm{H} = v\mathrm{K}$;

$$\therefore \frac{u}{v} = \frac{\mathrm{K}}{\mathrm{H}}.$$

64. When a fluid is in motion and acted on by any forces, to determine the effective accelerating force in the direction of its motion at any point.

Draw the curve $APQR$ (fig. 27) so that a tangent to it at any point may be in the direction of the motion of the fluid at that point. The motion of the fluid will not be altered if we suppose a portion of it, of the form of a very small cylinder having PQ for its axis, to be enclosed in a perfectly flexible and extensible envelope without weight.

Let p be the pressure of the fluid at P, S the accelerating force at P resolved in the direction of a tangent to APR at P, $AP = s$; then $p + d_s p . PQ$ will ultimately be the pressure at Q; and if ρ be the density of the fluid at P, κ the area of the base of the cylinder PQ; the mass of $PQ = \rho\kappa . PQ$, and the moving force on PQ in the direction PQ

$= S.(\text{mass } PQ) + \text{pressure on the end } P - \text{pressure on the end } Q$

$$= S\rho\kappa . PQ - \kappa d_s p . PQ;$$

therefore the effective accelerating force on the fluid at P in the direction PQ

$$= S - \frac{1}{\rho} d_s p.$$

65. To find the relation between the pressure and the velocity at P, when the velocity at any point is independent of the time.

Let v be the velocity at P, then $v d_s v$ = the effective accelerating force at P in the direction PQ;

$$\therefore\ v d_s v = S - \frac{1}{\rho} d_s p.$$

When the fluid is non-elastic

$$\tfrac{1}{2} v^2 + \frac{1}{\rho} p = \int_s S.$$

When the fluid is elastic, $p = \mu\rho$; $\therefore\ v d_s v = S - \frac{\mu}{p} d_s p$;

$$\therefore\ \tfrac{1}{2} v^2 + \mu \log_e p = \int_s S.$$

$\int_s S = \frac{1}{2} V^2 + C$, where V is the velocity acquired by a point acted on by the same forces as the fluid, in moving from A to P in a tube AP.

If the fluid be acted on by gravity only, and if z be the depth of P below a given horizontal plane, and γ the angle which PQ makes with a vertical through P, $S = g \cos\gamma$. But $\cos\gamma = d_s z$, therefore $S = g d_s z$, $\int_s S = gz + C$; hence, when the fluid is non-elastic,

$$\tfrac{1}{2} v^2 + \frac{1}{\rho} p = gz + C;$$

and when the fluid is elastic

$$\tfrac{1}{2} v^2 + \mu \log_e p = gz + C.$$

66. To find the relation between the pressure and the velocity at P, when the velocity depends upon the time as well as the position of P.

Let v be the velocity at P at the time t, PR the space described by a particle of the fluid in the very small time δt, therefore $PR = v\delta t$, $v + \delta v$ the velocity at R. v is a function of s and t;

$$\therefore \ \delta v = d_t v . \delta t + d_s v . v\delta t, \text{ ultimately.}$$

$$\text{But } \delta v = (S - \frac{1}{\rho} d_s p)\, \delta t, \text{ ultimately;}$$

$$\therefore \ d_t v + v d_s v = S - \frac{1}{\rho} d_s p.$$

67. Equation of continuity. Let PQ (fig. 27*) be the axis of the cylinder of fluid at the time t, $P'Q'$ its axis at the time $t + \delta t$; v the velocity of P, therefore $v + d_s v . PQ$ will be the velocity of Q. We have

$$PP' = v\delta t, \ \ QQ' = (v + d_s v . PQ)\, \delta t,$$

$$P'Q' = PQ + QQ' - PP' = (1 + d_s v \delta t)\, PQ.$$

Let κ, κ' be the bases of the cylinders PQ, $P'Q'$; ρ, ρ' the densities at P, P'. Then

$$\kappa' = \kappa + d_t \kappa . \delta t + d_s \kappa . v\delta t,$$

$$\rho' = \rho + d_t \rho \delta t + d_s \rho . v\delta t.$$

But mass PQ = mass $P'Q'$;

$$\therefore \ \rho\kappa . PQ = \rho'\kappa' . P'Q'.$$

Whence, substituting the values of ρ', κ', $P'Q'$ found above, and neglecting the powers of δt,

$$0 = d_t(\kappa\rho) + d_s(\kappa\rho v).$$

κ is a function of s and t depending upon the directions of the motion of the fluid immediately surrounding P. Thus when the motion of the fluid about P is in parallel lines, κ is constant. When the motion of the fluid at any instant is in straight lines which ultimately intersect in a point at the distance $s - a$ from P, κ will be proportional to $(s - a)^2$ or $(a - s)^2$ according as the motion is from or towards the point of intersection. The most general case is that in which the fluid moves in straight lines, which, like normals drawn from the boundary of a small portion of a curve surface, ultimately intersect in two lines perpendicular to each other in different planes at the distances $s-a$, $s-b$ from P. κ will be proportional to $(s-a)(s-b)$ when the fluid moves from both lines, and to $(a-s)(b-s)$ when it moves towards both lines; it will be proportional to $(s-a)(b-s)$ when the fluid moves from the former line towards the latter.

68. When ρ is constant, the equations obtained in (66) and (67) serve to determine p and v. When ρ is a function of p, we must substitute for ρ its value in terms of p in the equation of (66). From the equation thus obtained and the equation of continuity p and v are then to be found.

69. To find the velocity with which an incompressible fluid acted on by gravity issues through an indefinitely small orifice in the vessel containing it. Let K (fig. 28) be the orifice. Draw KH vertical meeting the plane of the surface of the fluid in H. Let $KH = h$; u the velocity of the fluid at K; p, v the pressure and velocity at any point P in the fluid, z the depth of P below the surface of the fluid; Π the pressure of the atmosphere. Then since the orifice is indefinitely small, the velocity at any point is very nearly independent of the time; $\therefore$ (65)

$$\tfrac{1}{2}v^2 + \frac{1}{\rho}p = gz + C.$$

At the surface of the fluid $z = 0$, $p = \Pi$, and $v = 0$; (for (64) velocity at the surface : u = area orifice : area surface, and the area of the orifice vanishes compared with the area of the surface of the fluid, therefore the velocity at the surface = 0.) $\therefore \frac{1}{\rho}\Pi = C$. At K, $z = h$, $p = \Pi$, since at the orifice the fluid is in contact with the atmosphere, $v = u$;

$$\therefore \tfrac{1}{2}u^2 + \frac{1}{\rho}\Pi = gh + C;$$

$$\therefore \tfrac{1}{2}u^2 = gh.$$

Or, the velocity of the issuing fluid is equal to the velocity acquired by a heavy body in falling down HK.

If M be the pressure of the fluid at any point in a horizontal plane meeting KH in H, Π the pressure of the atmosphere at K;

$$\frac{1}{\rho}\mathrm{M} = C, \text{ and } \tfrac{1}{2}u^2 + \frac{1}{\rho}\Pi = gh + C;$$

$$\therefore\ \tfrac{1}{2}u^2 = gh + \frac{1}{\rho}(\mathrm{M} - \Pi).$$

When the fluid issues through an orifice in a thin plate, it does not acquire its greatest velocity till it reaches a point at a small distance from the orifice. This part of the stream is called the "vena contracta," on account of the contraction of the stream resulting from its increased velocity. The area of a section of the "vena contracta" is equal to about $\frac{5}{8}$ of the area of the orifice.

70. Let KT be the direction of the issuing fluid; HT the intersection of the plane HKT and the plane of the surface of the fluid; Kx parallel to HT; $TKx = a$. Then each drop of the issuing stream being projected in the direction KT with the velocity acquired by a heavy body in falling down HK, and being acted on by gravity, will describe a parabola having HT for its directrix.

The equation to the curve described by the issuing stream is

$$y = x.\tan a - \frac{g}{2u^2}x^2(\text{secant } a)^2: \text{ or, } y = x.\tan a - \frac{x^2}{4h}(\text{secant } a)^2.$$

The velocity of the issuing fluid may be deduced from observed values of the angle TKx, and of the range of the stream on a horizontal plane at a given depth below the orifice. The value of the velocity determined in this manner is found to coincide very accurately with its theoretical value.

71. To find the time of emptying a vessel through a very small orifice.

Let κ be the area of the effective orifice, or of the section of the "vena contracta," x the depth of the orifice below the surface, X the area of the surface, u the velocity at the "vena contracta" at the end of the time t from the beginning of the motion. Then $-d_t x$ will be the velocity of the descending surface; and (63) $(-d_t x).\text{X} = u\kappa$; also if κ be very small compared with X, $u = \sqrt{(2gx)}$; $\therefore\ -d_t x.\text{X} = \kappa\sqrt{(2gx)}$. But $d_t x . d_x t = 1$;

$$\therefore\ \sqrt{(2g)}\,\kappa . d_x t + \frac{\text{X}}{\sqrt{(x)}} = 0.$$

Whence, since X is known in terms of x from the form of the vessel, by integration we can find the time in which the surface descends from any given altitude above the orifice to any given inferior altitude, and, consequently, the whole time of emptying.

It appears from experiment, that when the orifice is immersed in fluid, the quantity of fluid discharged in a given time, is the same as when the discharge takes place into air; the perpendicular distance between the surfaces of the fluids in the former case, being equal to the depth of the orifice below the surface in the latter.

72. An incompressible fluid acted on by gravity issues through a finite orifice in the horizontal base of the vessel in which it is contained; to determine its motion.

Draw AK (fig. 29) vertical meeting the plane of the orifice in K. At the time t let the surface of the fluid meet AK in H; and let p, v be the pressure and velocity at any point P in AK; $AK = c$; $AH = x$; $AP = z$; K the area of the orifice; X the area of the surface of the fluid; Z the area of a horizontal section of the vessel through P; Π the pressure of the atmosphere. Then, since the fluid is acted on by gravity only (65), $S = g d_s z$, therefore the equation found in (66) becomes

$$d_t v + v d_s v = g d_s z - \frac{1}{\rho} d_s p\,;$$

$$\therefore\ \textstyle\int_s d_t v + \frac{1}{2}v^2 = gz - \frac{1}{\rho}p.$$

If we suppose the motion of the fluid to be vertical, and the velocity, at a given instant, the same at all points in the same horizontal section, $d_z s = 1$, and $\text{Z}v = \text{K}u$. Now Z is independent of t, and $d_t u$ is independent of z; $\therefore\ \text{Z} d_t v = \text{K} d_t u$.

$$\textstyle\int_s d_t v = \int_z d_t v . d_z s = \int_z d_t v = \int_z \frac{\text{K}}{\text{Z}} d_t u = \text{K} . d_t u \int_z \frac{1}{\text{Z}}\,.$$

Hence the equation $\int_s d_t v + \frac{1}{2}v^2 = gs - \frac{1}{\rho}p$ becomes

$$\text{K}\, d_t u \int_s \frac{1}{Z} + \frac{1}{2}\frac{\text{K}^2}{Z^2}u^2 = gs - \frac{1}{\rho}p. \quad (1)$$

At H, $s = x$, $Z = \text{X}$, $p = \Pi$; at K, $s = c$, $Z = \text{K}$, $p = \Pi$;

$$\therefore\ \text{K}.d_t u.\int_x^c \frac{1}{Z} + \frac{1}{2}\left(1 - \frac{\text{K}^2}{\text{X}^2}\right)u^2 = g(c - x). \quad (2).$$

The velocity of the surface of the fluid is $d_t x$, and $d_t u = d_x u.d_t x$, therefore $d_t x$ being known in terms of x and u, for $-d_t x.\text{X} = \text{K}u$, (2) will give u and x in terms of t, and then p may be obtained from (1).

When the issuing stream is contracted, the section of the "vena contracta" must be considered as the orifice.

If the pressure at $H = \text{M}$, and the pressure at the orifice $= \Pi$,

$$\text{K}d_t u\int_x^c \frac{1}{Z} + \frac{1}{2}\left(1 - \frac{\text{K}^2}{\text{X}^2}\right)u^2 = g(c - x) + \frac{1}{\rho}(\text{M} - \Pi).$$

73. When the vessel is continually supplied with fluid, so that the surface of the fluid remains stationary, x, and $\int_x^c \frac{1}{Z}$ are constant. Hence, if $\text{K}\lambda\int_x^c \frac{1}{Z} = \{2g(c - x)\}^{\frac{1}{2}}.\left(1 - \frac{\text{K}^2}{\text{X}^2}\right)^{\frac{1}{2}}$, (Hymers' Integral Calculus, 21),

$$\{2g(c-x)\}^{\frac{1}{2}} - u\left(1 - \frac{\text{K}^2}{\text{X}^2}\right)^{\frac{1}{2}} = \left[\{2g(c-x)\}^{\frac{1}{2}} + u\left(1 - \frac{\text{K}^2}{\text{X}^2}\right)^{\frac{1}{2}}\right]\epsilon^{-\lambda t}.$$

When K is small, and t finite, λt is very large, and therefore $\epsilon^{-\lambda t}$ is very small. Consequently at the end of a finite time from the beginning of the motion,

$$u^2\left(1 - \frac{\text{K}^2}{\text{X}^2}\right) = 2g(c - x) \text{ very nearly.}$$

When the velocity of the fluid at a given point is independent of the time, $d_t u = 0$;

$$\therefore\ u^2\left(1 - \frac{\text{K}^2}{\text{X}^2}\right) = 2g(c - x).$$

74. When the waste of the fluid is not supplied, $X.d_t x = Ku$, and equation (2) becomes

$$\frac{K^2}{X}.u\,d_t u \int_x^c \frac{1}{Z} + \tfrac{1}{2}\left(1 - \frac{K^2}{X^2}\right) u^2 = g\,(c - x)\,;$$

$$\therefore \text{ if } \int_x^c \frac{1}{Z} = N, \text{ (Hymers' Differential Equations, 15)}$$

$$\tfrac{1}{2}K^2u^2 = \epsilon^{-\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}} . \int_x g\,\frac{X}{N}\,(c - x)\,\epsilon^{\int_x\left(\frac{1}{K^2}-\frac{1}{X^2}\right)\frac{X}{N}}.$$

75. An incompressible fluid acted on by gravity flows through a tube; to determine the motion of the fluid.

Let APK (fig. 30) be the axis of the tube; and, at the time t, let p be the pressure, and v the velocity at P; $AP = s$; z the depth of P below a horizontal plane through A; S the area of a section of the tube at P; u the velocity at any point K; K the area of a section of the tube at K.

Then since the fluid is acted on by gravity only, (65).

$$d_t v + v\,d_s v = g\,d_s z - \frac{1}{\rho}\,d_s p\,; \quad \therefore \int_s d_t v + \tfrac{1}{2}v^2 = gz - \frac{1}{\rho}\,p.$$

$Sv = Ku$, and S is independent of t; therefore $S.d_t v = K.d_t u$, and $d_t u$ is independent of s;

$$\therefore \int_s d_t v = K\,d_t u \int_s \frac{1}{S}\,; \quad \therefore K\,d_t u \int_s \frac{1}{S} + \tfrac{1}{2}\,\frac{K^2}{u^2}\,u^2 = gz - \frac{1}{\rho}\,p. \quad (1)$$

Let H, L be the extremities of the column of fluid; $AH = s_{,}$, $AL = s_{,,}$; $S_{,}$, $S_{,,}$ the areas of sections of the tube at H, L; $z_{,}$, $z_{,,}$ the depths of H, L below a horizontal plane through A. Then

$$K.d_t u \int_{s_{,}}^{s_{,,}} \frac{1}{S} + \tfrac{1}{2}\left(\frac{1}{S_{,,}^2} - \frac{1}{S_{,}^2}\right) K^2u^2 = g\,(z_{,,} - z_{,}). \quad (2)$$

When the quantity of fluid in the tube is constant, let V be its volume, then

$$V = \int_{s_{,}}^{s_{,,}} S\,; \quad S_{,}.d_t s_{,} = Ku\,; \quad S_{,,}.d_t s_{,,} = Ku.$$

From these equations, equation (2), and the equations to the axis of the tube, we may obtain $s_{,}$, $s_{,,}$, u.

If K be the extremity of the tube, $AK = c$, a the depth of K below a horizontal plane through A;

$$\text{K} d_t u \int_{s_,}^{c} \frac{1}{S} + \frac{1}{2}\left(1 - \frac{\text{K}^2}{\text{S}_,^2}\right) u^2 = g(a - z_,) \quad (3);$$

$\text{S}_, . d_t s_, = \text{K} u$; and when $d_t s_,$ is known in terms of $s_,$ and u, $s_,$, u may be found from (3) and then p may be found from (1).

76. To determine the velocity with which a small disturbance is propagated along a horizontal column of fluid.

Let AP (fig. 31) be a horizontal tube filled with fluid, the equilibrium of which has been slightly disturbed; P, Q discs serving to separate the fluid between P and Q, from the rest of the fluid in the tube, without impeding its motion; κ the area of a section of the tube; x, and $x + \delta x$ the distances of P and Q from A, and ρ the density of the fluid, before its equilibrium was disturbed. At the time t let $AP = y$, and the pressure at $P = p$; therefore, ultimately, $AQ = y + d_x y . \delta x$, and the pressure at $Q = p + d_x p . \delta x$.

The moving force on the cylinder of fluid PQ in the direction AP = pressure on the end P − pressure on the end Q $= -\kappa d_x p . \delta x$; the mass of the fluid in $PQ = \kappa \rho . \delta x$; therefore effective acc^e. force on PQ in the direction $AP = -\frac{1}{\rho} d_x p$;

$$\therefore\ d_t^2 y = -\frac{1}{\rho} d_x p.$$

(1) Let the fluid be air, T its temperature and ρ its density; $p = \mu(1 + \text{ET})\rho$ the equation expressing the relation between the pressure and density of the fluid at the temperature T when at rest. T$'$ the temperature, ρ' the density of the fluid at P, $\omega\left(1 - \frac{\rho}{\rho'}\right)$ the number of degrees by which the temperature of a given mass of the fluid is increased when its density is suddenly changed from ρ to ρ', therefore

$$\text{T}' - \text{T} = \omega\left(1 - \frac{\rho}{\rho'}\right).$$

The volume of $PQ = \kappa d_x y . \delta x$, therefore the mass of the fluid in $PQ = \kappa \rho' d_x y . \delta x$; but the mass of the fluid in $PQ = \kappa \rho . \delta x$, therefore $\rho' d_x y = \rho$. If p be the pressure at P,

$$p = \mu (1 + \text{ET}') \rho' = \mu \{1 + \text{ET} + \text{E} (\text{T}' - \text{T})\} \rho'$$

$$= \mu \left(1 + \frac{\text{E}\omega}{1 + \text{ET}}\right) (1 + \text{ET}) \rho' - \mu \text{E} \omega \rho$$

$$= \mu \text{K} (1 + \text{ET}) \rho \frac{1}{d_x y} - \mu \text{E} \omega \rho, \text{ where } \text{K} = 1 + \frac{\text{E}\omega}{1 + \text{ET}};$$

$$\therefore \frac{1}{\rho} d_x p = - \mu \text{K} (1 + \text{ET}) \frac{d_x^2 y}{(d_x y)^2}.$$

Since the disturbance is small, ρ and ρ' are very nearly equal, therefore $d_x y = 1$, very nearly;

$$\therefore \frac{1}{\rho} d_x p = - \mu \text{K} (1 + \text{ET}) d_x^2 y;$$

$$\therefore d_t^2 y = \mu \text{K} (1 + \text{ET}) d_x^2 y.$$

(2) Let the fluid be of the kind denominated non-elastic, or liquid. The increment of the density of a liquid under a moderate pressure is found to be proportional to the pressure. Let, therefore, $\rho + \frac{p}{\mu}$ be the density of the liquid under the pressure p. Then $\kappa \left(\rho + \frac{p}{\mu}\right) d_x y . \delta x =$ mass of $PQ = \kappa \rho . \delta x$. Since the disturbance is very small, p is very small compared with $\mu \rho$, therefore $\mu \rho d_x y = \mu \rho - p$ very nearly;

$$\therefore \mu \rho d_x^2 y = - d_x p; \quad \therefore d_t^2 y = \mu d_x^2 y.$$

In liquids, the heat developed by compression is nearly insensible.

(3) Let AP be a rod vibrating longitudinally, $\frac{ca}{\mu \rho}$ the quantity by which the rod is shortened, or lengthened, when it is compressed longitudinally, or extended, by a pressure κp, the original length of the rod being c. Then $d_x y . \delta x =$ length of PQ

$$= \left(1 - \frac{p}{\mu \rho}\right) . \delta x; \quad \therefore \mu \rho d_x^2 y = - d_x p;$$

$$\therefore\ d_t^2 y = \mu d_x^2 y.$$

In consequence of the variation of the thickness of a rod, when it is compressed longitudinally, or extended, the variation of its length (Pouillet, Elémens de Physique, 463) is 1,5 of what it would be if its thickness were invariable. It is the variation of the length of the rod, on the latter supposition, that is to be used in deducing the value of μ.

77. The equation of the motion of the disturbance is of the form $d_t^2 y = a^2 d_x^2 y$. The integral of this equation may be made to depend on that of $d_t z = a d_x z$, which we know to be $z = \phi(x + at)$, in the following manner:

$$d_t d_t y = a^2 d_x d_x y, \quad d_t d_x y = d_x d_t y;$$

$$\therefore\ d_t(a d_x y + d_t y) = a d_x (a d_x y + d_t y),$$

$$d_t(a d_x y - d_t y) = - a d_x (a d_x y - d_t y);$$

$$\therefore\ a d_x y + d_t y = \phi(x + at),$$

$$a d_x y - d_t y = \psi(x - at);$$

$$\therefore\ 2 a d_x y = \phi(x + at) + \psi(x - at),$$

$$2 d_t y = \phi(x + at) - \psi(x - at);$$

$$\text{and } y = F(x + at) + f(x - at).$$

78. Let the initial disturbance extend through a very small space $2a$, that is, from $-a$ to a; then at the beginning of the motion, or when $t = 0$, $a d_x y + d_t y = \phi x$, $a d_x y - d_t y = \psi x$; and the fluid at any point distant from A by a quantity greater than a will be at rest, therefore $d_x y = 1$, $d_t y = 0$; and therefore $\phi x = a$, $\psi x = a$, as long as x does not lie between $-a$ and a. Therefore $\phi(x + at) = a$, except when $x + at$ lies between $-a$ and a; and $\psi(x - at) = a$, except when $x - at$ lies between $-a$ and a; hence $d_t y = 0$, except when one of the quantities $x + at$, $x - at$ lies between $-a$ and a. When $x - at$ is less than $-a$, or greater than a, $x + at$ is greater than a; therefore if P, R be any two points in AP, the fluid at P will remain at rest till $AP - at = a$, or till the time $(AP - a) \div a$, it will then begin to move, and will return to a state of permanent rest when $AP - at = -a$, or at the time $(AP + a) \div a$. In like manner,

the fluid at R will begin to move at the time $(AR - a) \div a$, and will return to a state of rest at the time $(AR + a) \div a$. Hence the fluid at R will begin to move $(PR \div a)$ later than the fluid at P; therefore the velocity with which the disturbance is propagated $= \{PR \div (PR \div a)\} = a$.

79. The motion of a small disturbance propagated along a horizontal tube might have been determined by means of the equations in (66) and (67). We shall employ these equations in the solution of the following problem:

To determine the motion of a small disturbance propagated in air symmetrically through a solid angle.

Let v be the velocity, ρ the density, p the pressure of the air at the distance s from the vertex of the solid angle at the time t; S the resolved part of the force estimated in the direction in which s is measured.

The motion of the air at any point is directed from or towards the vertex of the solid angle, therefore $\kappa \propto s^2$. The variation of ρ is very small compared with that of v, therefore we may substitute $\rho d_s(\kappa v)$ for $d_s(\rho\kappa v)$; s is independent of t, therefore $d_t(s^2\rho) = s^2 d_t\rho$. Hence the equation of continuity (67) becomes $s^2 d_t\rho + \rho d_s(s^2 v) = 0$.

Since the velocity of the air is small, $v d_s v$ vanishes compared with $d_t v$, therefore the equation in (66) becomes

$$d_t v = S - \frac{1}{\rho} d_s p.$$

But, if T be the temperature of the air when at rest, $d_s p = a^2 d_s \rho$, where $a^2 = \mu K(1 + ET)$;

$$\therefore\ d_t v = S - \frac{a^2}{\rho} d_s \rho\,; \quad \therefore\ \textstyle\int_s d_t v = \int_s S - a^2 \log_e \rho,$$

$$d_t \textstyle\int_s d_t v = -a^2 \frac{1}{\rho} d_t \rho = \frac{a^2}{s^2} d_s(s^2 v). \quad \text{If } d_s\phi = v,$$

$$\frac{1}{s^2} d_s(s^2 v) = 2\frac{v}{s} + d_s v = \frac{1}{s}(2 d_s\phi + s d_s^2\phi) = \frac{1}{s} d_s^2(s\phi),$$

$$d_t \textstyle\int_s d_t v = d_t^2\phi = \frac{1}{s} d_t^2(s\phi)\,; \quad \therefore\ d_t^2(s\phi) = a^2 d_s^2(s\phi)\,;$$

$$\therefore \ (77) \quad s\phi = F(s + at) + f(s - at).$$

Whence, differentiating with respect to s, and observing that $d_s\phi = v$,

$$v = \frac{1}{s}\{d_sF(s+at) + d_sf(s-at)\} - \frac{1}{s^2}\{F(s+at) + f(s-at)\}.$$

It may be shewn exactly as in (78) that a is the velocity with which the disturbance is propagated.

80. Sound is produced by a repetition of such disturbances. The velocity of sound in any medium is the velocity with which a small disturbance is propagated through it. Hence the velocity of sound in air at the temperature T

$$= \sqrt{\{\mu K (1 + ET)\}}.$$

If ρ be the density of dry air at the temperature T under the pressure Π, $\rho_{\prime}$ the density of air mixed with vapour, the pressure of which is Υ, at the same temperature and under the same pressure,

$$\rho_{\prime} = \left(1 - 0{,}375\frac{\Upsilon}{\Pi}\right)\rho \quad (41); \ \therefore \ \Pi = \mu(1 + ET)\rho$$

$$= \{\mu(1+ET)\rho_{\prime}\} \div \left(1 - 0{,}375\frac{\Upsilon}{\Pi}\right) = \{\mu(1+ET)\rho_{\prime}\}.\left(1 + 0{,}375\frac{\Upsilon}{\Pi}\right)$$

very nearly.

But the velocity of sound in a medium the pressure and density of which are Π and $\rho_{\prime}$ respectively $= (\Pi K) \div \rho_{\prime}$. Hence the velocity of sound in the mixture of air and vapour

$$= \sqrt{\left\{\mu K (1 + ET)\left(1 + 0{,}375\frac{\Upsilon}{\Pi}\right)\right\}}.$$

According to the best of the observations made by MM. Moll, Vanbeek, and Kuytenbrower, (Phil. Trans. 1824, 1830), sound travels 17669,28 mètres in 51,9873 seconds of time, the temperature of the air being 11,01° C, and its pressure and the pressure of the vapour mixed with it being equal to the pressures of 0,74618, 0,00889 mètres of mercury respectively. From these data it appears that sound travels at the rate of

$$1090{,}8\sqrt{\left\{(1 + 0{,}003665\,T)\left(1 + 0{,}375\frac{\Upsilon}{\Pi}\right)\right\}}$$

feet in one second of time, in atmospheric air at the temperature T, Π being the pressure of the air and Υ the pressure of the vapour contained in it.

Attempts have been made to determine K by observing the change of temperature which a given mass of air undergoes when suddenly compressed or dilated. Its value obtained by this method is 1,3748, according to Gay Lussac and Welter. It may be deduced with greater accuracy from the observed velocity of sound. We have seen that $\sqrt{(\mu K)} = 1090{,}8$ feet, and that $\sqrt{(\mu)} = 916{,}188$ feet, whence $K = 1{,}4175$.

The experiments of Oersted shew, that when water is pressed by a column of water 33,14 feet high, its density is increased by 0,0000461 of its original density, the force of gravity being 32,21 feet;

$$\therefore \mu = (32{,}81)\ (32{,}21) \div (0{,}0000461), \quad \sqrt{(\mu)} = 4860.$$

The velocity observed by MM. Colladon and Sturm was 4708 feet in one second.

81. To express the equations of the motion of a fluid in terms of rectangular co-ordinates.

Let PQ (fig. 21) be a parallelopiped of the fluid, having its edges PL, PM, PN parallel to the axes of co-ordinates, and which we may suppose to be enclosed in a perfectly flexible and extensible envelope. Let P', Q', L', M', N' be the places of P, Q, L, M, N at the time $t + \delta t$. x, y, z the co-ordinates of P; $x + \delta x$, $y + \delta y$, $z + \delta z$ those of Q; u, v, w the components of the velocity of P parallel to the axes of x, y, z. p the pressure, ρ the density of the fluid, at P at the time t, and X, Y, Z the components of the resultant of the forces that act upon the fluid at P resolved parallel to the axes.

The pressure at L will be $p + d_x p\, \delta x$ ult. The moving force on PQ in a direction parallel to OX

$$= \text{press. on } NM + (\text{mass } PQ)\ X - \text{press. on } QL$$

$$= \delta x\, \delta y\, \delta z . (\rho X - d_x p).$$

But mass $PQ = \rho\delta x\delta y\delta z$; therefore the effective accelerating force on PQ parallel to OX

$$= X - \frac{1}{\rho} d_x p.$$

In like manner the effective accelerating force on PQ in a direction parallel to OY

$$= Y - \frac{1}{\rho} d_y p.$$

And the effective accelerating force parallel to OZ

$$= Z - \frac{1}{\rho} d_z p.$$

u, v, w, are functions of t, x, y, z; the co-ordinates of P' are $x + u\delta t$, $y + v\delta t$, $z + w\delta t$. Therefore, at the time $t + \delta t$, when P arrives at P', the component of the velocity of the fluid at P' parallel to OX will be

$$u + d_t u\delta t + d_x u . u\delta t + d_y u . v\delta t + d_z u . w\delta t, \text{ ultimately.}$$

But the increment of the velocity estimated in a given direction in the time δt is equal to the product of the effective accelerating force estimated in the same direction multiplied by δt. Hence

$$d_t u + u d_x u + v d_y u + w d_z u = X - \frac{1}{\rho} d_x p.$$

In like manner

$$d_t v + u d_x v + v d_y v + w d_z v = Y - \frac{1}{\rho} d_y p.$$

And

$$d_t w + u d_x w + v d_y w + w d_z w = Z - \frac{1}{\rho} d_z p.$$

82. A fourth equation, called "the equation of continuity," is furnished by the condition that the mass of the fluid contained within the figure $P'Q'$ must be equal to that of the fluid contained within the figure PQ.

The co-ordinates of L', which may be deduced by substituting $x + \delta x$ for x in those of P', are ultimately

$$x + \delta x + u\delta t + d_x u\,\delta x\,\delta t,$$

$$y + v\delta t + d_x v\,\delta x\,\delta t,$$

$$z + w\delta t + d_x w\,\delta x\,\delta t.$$

By comparing the co-ordinates of P', L' with those of P, L it appears that $P'L'$, Ox are nearly parallel, and that

$$P'L' = \delta x\,(1 + d_x u\delta t).$$

In like manner $P'M'$, Oy are nearly parallel, and

$$P'M' = \delta y\,(1 + d_y v\delta t);$$

and $P'N'$, Oz nearly parallel, and

$$P'N' = \delta z\,(1 + d_z w\delta t).$$

In the same way $P'L'$, $P'M'$, $P'N'$ are found to be ultimately parallel and equal to the edges of $P'Q'$, which are respectively opposite to them. Hence

$$\text{vol.}\,P'Q' = P'L'.P'M'.P'N' = (1 + d_x u\delta t + d_y v\delta t + d_z w\delta t)\,\delta x\,\delta y\,\delta z.$$

$$\text{Density at } P' = \rho + d_t\rho\,\delta t + d_x\rho.u\delta t + d_y\rho.v\delta t + d_z\rho.w\delta t.$$

Mass $P'Q'$ = (vol. $P'Q'$) (density at P'). But mass $P'Q'$ = mass PQ. Whence

$$d_t\rho + ud_x\rho + vd_y\rho + wd_z\rho + \rho d_x u + \rho d_y v + \rho d_z w = 0,$$

or

$$d_t\rho + d_x(\rho u) + d_y(\rho v) + d_z(\rho w) = 0.$$

When the fluid is compressible ρ is known in terms of p. When the fluid is incompressible ρ is invariable, and the preceding equation reduces itself to

$$d_x u + d_y v + d_z w = 0.$$

In either case, therefore, we have four equations to determine the four unknown quantities p, u, v, w, in terms of t, x, y, z.

83. If a quantity ϕ can be found such that

$$d_x\phi = u, \quad d_y\phi = v, \quad d_z\phi = w,$$

a quantity V such that

$$d_x V = X,\quad d_y V = Y,\quad d_z V = Z,$$

and a quantity P such that

$$d_x P = \frac{1}{\rho} d_x p,\quad d_y P = \frac{1}{\rho} d_y p,\quad d_z P = \frac{1}{\rho} d_z p,$$

the equation

$$d_t\phi + \tfrac{1}{2}\{(d_x\phi)^2 + (d_y\phi)^2 + (d_z\phi)^2\} = V - P$$

will satisfy each of the three equations (81)

This is in fact nothing more than the equation

$$d_t v + v d_s v = S - \frac{1}{\rho} d_s p,$$

which we obtained in (66) by a simpler process. For if we integrate each term with respect to s it becomes

$$\int_s d_t v + \tfrac{1}{2} v^2 = \int_s S - \int_s \frac{1}{\rho} d_s p\,;$$

and it may be easily shewn that

$$\int_s S = V,\quad \int_s \frac{1}{\rho} d_s p = P;\quad \int_s d_t v = d_t\phi,\quad v^2 = (d_x\phi)^2 + (d_y\phi)^2 + (d_z\phi)^2.$$

The equation of continuity expressed in terms of ϕ becomes

$$d_t\rho + d_x(\rho d_x\phi) + d_y(\rho d_y\phi) + d_z(\rho d_z\phi) = 0.$$

When the fluid is incompressible,

$$d_x^2\phi + d_y^2\phi + d_z^2\phi = 0.$$

SECTION VI.

ON RESISTANCES.

ART. 84. THE resistance of a fluid on a solid moving in it, is the resultant of the excess of the pressure of the fluid on the solid in motion, above the pressure of the fluid on the solid at rest.

Let APB (fig. 32) be a solid, moving in a fluid with the velocity V in the direction BA. Now if we communicate to the fluid and the solid a velocity V in the direction AB, the pressure of the fluid on APB will not be altered; and the solid will be at rest in a fluid moving in the direction AB with a velocity V. Hence the force with which a fluid in motion impels a solid immersed in it, is equal to the resistance of a stagnant fluid on a solid in motion, the velocity of the fluid in one case being equal to the velocity of the solid in the other. So also, when both the solid and fluid are in motion, the resistance on the solid is equal to the force with which the solid at rest would be impelled by a stream moving with the relative velocity of the fluid and solid.

Let the cylindrical surface generated by a straight line parallel to AB touch the solid in the curve PQR. The pressure on the surface $RAPQ$, will upon the whole be greater, and the pressure upon $PBQR$, less, than when the solid and fluid are relatively at rest. In the following Articles we shall consider that part only of the resistance, which arises from the increased pressure on $RAPQ$.

It must be observed that the following theory of resistances, which is the same as that given in the Note, page 188, of Mr Moseley's Hydrostatics, is very imperfect, and that it is useless to expect any close agreement between the results deduced from it, and those obtained by experiment.

85. To find the force with which a stream impels a plane, the plane being perpendicular to the direction of the stream.

Let P (fig. 33) be a point in the plane; EP the direction of the stream, perpendicular to the plane; p' the pressure at P; p the pressure of the fluid at P before the plane was immersed, or, the pressure of the fluid at the point P in a plane moving with the same velocity, and in the same direction as the fluid; ρ the density of the fluid; K the area of the plane.

Then $p' - p$ will be the resistance of a unit of the plane at P. Now (65) $\frac{1}{2}v^2 = \int_s S - \frac{1}{\rho}p$; and after the plane is immersed, the velocity at $P = 0$; $\therefore 0 = \int_s S - \frac{1}{\rho}p'$; $\therefore p' - p = \frac{1}{2}\rho v^2$, and the resistance on the plane $= (p' - p)\,\text{K} = \frac{1}{2}\rho v^2\text{K}$.

$\frac{1}{2}\rho v^2\text{K}$ is the weight of a column of fluid having the given plane for its base, and the altitude of which is the space due to the velocity of the fluid.

If the plane be made to move in a direction perpendicular to EP, it is manifest that the force with which the stream impels the plane will not be altered. Hence the force with which a given fluid impels a given plane, depends only on that part of the relative velocity of the fluid and plane which is perpendicular to the plane. Also, since the resistance, or impelling force of the fluid, arises from the pressure of the fluid on the plane, it must act in a direction perpendicular to the plane.

86. A stream impinges obliquely on a plane; to find the force with which the stream impels the plane.

Let P (fig. 33) be a point in the plane; AP the direction of the stream; EP perpendicular to the plane; v the velocity of the stream; R the resistance, or the force with which the stream impels the plane.

The velocity of the stream estimated in the direction EP

$$= v.\cos APE; \quad \therefore R = \tfrac{1}{2}\rho v^2.(\cos APE)^2\text{K}.$$

The resolved part of the impelling force estimated in the direction of the stream

$$= R.\cos APE = \tfrac{1}{2}\rho v^2.(\cos APE)^2\text{K}.$$

The resolved part of the impelling force estimated in a direction perpendicular to the stream, and in the plane APE,

$$= R.\sin APE = \tfrac{1}{2}\rho v^2.(\cos APE)^2.\sin APE.\text{K}.$$

87. A portion of a cylindrical surface having the curve BPC (fig. 34) for its base, and bounded by planes parallel and perpendicular to its generating straight line, is immersed in a stream flowing in the direction AE; to find the force with which the stream impels the cylinder in the directions AE and MA.

Draw AN perpendicular to AE; PM, QN parallel to AE; PE a normal to BP at P. Let a be the altitude of the cylinder, $MP = x$, $AM = y$, $MN = \delta y$, R the impelling force, or resistance, on that part of the cylinder which stands on BP, estimated in the direction AE, therefore ultimately $d_y R.\delta y$ = resistance on that part of the cylinder which stands on $PQ = \frac{1}{2}\rho v^2 (\cos AEP)^2 a.PQ = \frac{1}{2}\rho v^2 (\cos AEP)^2 a\,\delta y$,

$$\text{and } \tan AEP = -d_y x; \quad \therefore\ d_y R = \tfrac{1}{2}\rho v^2 \frac{a}{1 + (d_y x)^2},$$

$$\text{and } R = \tfrac{1}{2}\rho v^2 a \int_y \frac{1}{1 + (d_y x)^2}.$$

So also, if S be the resistance on the part BP of the cylinder, estimated in the direction MA,

$$d_y S.\delta y = \tfrac{1}{2}\rho v^2 (\cos AEP)^2 \sin AEP.a.PQ$$

$$= \tfrac{1}{2}\rho v^2.\cos AEP.\sin AEP.a.\delta y;$$

$$\therefore\ d_y S = \tfrac{1}{2}\rho v^2 a \frac{d_y x}{1 + (d_y x)^2}, \quad \text{and } S = \tfrac{1}{2}\rho v^2 a \int_y \frac{d_y x}{1 + (d_y x)^2}.$$

The integrals must be taken between limits corresponding to B and C.

88. A solid is generated by the revolution of the curve BPC (fig. 34) round AE; to find the force with which it is impelled by a stream moving in the direction AE.

Let R be the resistance on that part of the solid which is generated by the revolution of BP round AE; then, retaining the construction and notation of the preceding Article, we have ultimately, $d_y R \,.\, \delta y = \frac{1}{2} v^2 . (\cos PEA)^3 . 2\pi \,.\, AM \,.\, PQ$

$$= \rho v^2 \pi \frac{y \delta y}{1 + (d_y x)^2}; \quad \therefore \; d_y R = \pi \rho v^2 \frac{y}{1 + (d_y x)^2};$$

$$\therefore \; R = \rho v^2 \pi \int_y \frac{y}{1 + (d_y x)^2}, \text{ from } B \text{ to } C.$$

SECTION VII.

DESCRIPTION OF INSTRUMENTS. METHODS OF FINDING SPECIFIC GRAVITIES, &c.

Art. 89. Almost all bodies expand by heat, and contract by cold. This property furnishes the only known mode of comparing and recording the temperatures to which any body is exposed. The expansions of mercury, or air, combined with that of the glass vessel in which they are contained, are usually employed for this purpose.

90. The common mercurial thermometer is a glass tube of uniform bore, having a bulb at one end, which, with part of the tube, is filled with mercury; the other end is usually sealed, the space between it and the mercury being a vacuum.

To fill the thermometer with mercury, the air must be partly expelled from the bulb by holding it over the flame of a lamp, and then, the other end, which is open, immersed in mercury. As the bulb cools, the mercury will be forced into it by the pressure of the atmosphere. If a paper funnel be now tied round the open end, and filled with mercury; and the mercury in the bulb be heated till it boils, the remainder of the air will be driven out, and its place supplied by mercurial vapour: this condenses on cooling, and the mercury will descend from the funnel and fill the instrument completely. When it has cooled down nearly to the highest temperature intended to be measured by it, the open end must be sealed; and as it continues to cool, the mercury will descend, leaving a vacuum in the upper part of the tube.

91. To graduate a thermometer.

Let the bulb, and that part of the tube which is occupied by the mercury, be immersed in melting snow, and make a mark on the tube, opposite to the extremity of the column of mercury, when it is stationary. This is the freezing point. Next let the thermometer be surrounded by the vapour of water boiling under a given atmospheric pressure, and make a mark on the tube at the place where the extremity of the column of mercury rests, when it is stationary. This is the boiling point. The space between the freezing and boiling points, in the centigrade thermometer, is divided into 100 parts of equal volume, called degrees; the freezing point being called 0°, and the boiling point 100°. In Fahrenheit's thermometer the space is divided into 180 parts. The freezing point is marked 32°; and the boiling point 212°. In Reaumur's thermometer the freezing point is marked 0°, and the boiling point 80°.

92. The temperature of melting snow is found to be the same under all circumstances. The temperature of steam, however, varies with the atmospheric pressure. 100° of the centigrade thermometer denotes the temperature of steam, when the pressure of the atmosphere is equal to that of a column of mercury at 0°, 0,76 mètres, or 29,9218 inches high, at the mean level of the sea in lat. 45°. A variation of 1,045 inches in the height of the mercury in the barometer, occasions a change of 1° in the temperature of steam. The boiling point of Fahrenheit's scale has never been properly defined.

93. To compare the scales of two differently graduated thermometers.

Let A, B, X be any three points on a thermometer tube; a, b, x; a', b', x' the corresponding degrees in two different scales. Then, since the number of degrees contained between any two points on the tube is proportional to the capacity of the corresponding portion of the tube,

$$\frac{x-a}{b-a} = \frac{\text{vol. } AX}{\text{vol. } AB}, \quad \frac{x'-a'}{b'-a'} = \frac{\text{vol. } AX}{\text{vol. } AB};$$

$$\therefore \frac{x-a}{b-a} = \frac{x'-a'}{b'-a'}.$$

Let C° of the centigrade thermometer, F° of Fahrenheit's, R° of Reaumur's, denote the same temperature; then, if we drical bar of iron passing through a water-tight collar in the bottom of the hollow cylinder *AB*, has one end fastened into the cylinder *M*, while a ring at the other end serves to connect it with the pile or cable.

THE DIVING BELL. (Fig. 36.)

97. The diving bell is a chest, the weight of which is greater than that of the water it would contain, suspended by a rope with its mouth downwards. If the bell be lowered out of air into water in this position, the air contained in it will prevent the water from rising in the upper part of the bell, and thus enable persons to breathe at considerable depths below the surface of the water.

To find the space occupied by the air in the bell at any depth below the surface.

Let *BE* be the bell, draw *AM* vertical meeting the surface of the water on the outside of the bell in *A*, and the surface of the water within the bell in *M*; and let *h* be the altitude of a column of water the pressure of which is equal to that of the atmosphere (about 34 feet). When the bell was at the surface the air in it occupied the space *DECB*, under a pressure equal to that of a column of water the height of which is *h*; and the pressure at *M* is that of a column of water the height of which is $h + AM$;

$$\therefore (31.)\quad \frac{\text{vol. } BMC}{\text{vol. } DECB} = \frac{\text{atmospheric pressure}}{\text{pressure at } M} = \frac{h}{h + AM}.$$

The water may be almost wholly expelled from the interior of the bell by a supply of air from above, forced by an air-pump through a flexible tube terminating under the mouth of the bell. In this manner also the air is changed as often as it becomes unfit for respiration.

THE SYPHON. (Fig. 37.)

98. The syphon is a bent tube *ABC* open at both ends. Let the ends be closed, after filling it with fluid, and place it with one end in a bowl of the fluid with which it was filled, so

that the other end may be below the surface of the fluid in the bowl. Let the plane of the surface of the fluid in the bowl meet the legs of the syphon in *H*, *K*, the height of *B* above the surface of the fluid being less than the height of a column of the fluid, the pressure of which is equal to the atmospheric pressure Π. If the end *A* be opened, the pressure within the tube at *H* will be Π; and if the end *C* be opened, the pressure at *C* will be Π. Since the columns of fluid *HB*, *BK* would be in equilibrium if the pressures at their lower ends were equal, the columns *HB*, *BC* will not be in equilibrium, but the column *BC* which has the greater altitude will descend and run out at *C*, while the fluid in the bowl is forced up *AB* by the pressure of the atmosphere. And this will continue till the surface descends to the level of the highest end of the syphon.

The syphon will not act when the altitude of the highest part of it above the surface of the fluid in the bowl, is greater than the height of a column of the fluid the pressure of which is equal to that of the atmosphere. For on opening *A*, the fluid in *BA* will sink till its altitude is such that the pressure it exerts at *H*, becomes equal to the pressure of the atmosphere, leaving a vacuum at *B*.

The Common Pump. (Fig. 38.)

99. *AB*, *BC* are two hollow cylinders having a common axis; *C* the surface of the water into which the extremity of *BC* descends; *M* a piston capable of being moved up and down by a rod *MA*, and containing a valve opening upwards; *AB* the range of the piston; *B* a valve opening upwards; *D* a spout placed a little above *A*.

Suppose *M* to be at *B*, and the pump to be filled with air the pressure of which is equal to that of the atmosphere; and let *M* be elevated to *A*. Then, the air in *BC* will open the valve *B* and fill *AB*, and the pressure of the air in the pump being less when it occupies the space *ABC*, than when it occupied the space *BC*, the pressure of the atmosphere will force the water up *BC* till the pressure at *C* is the same as before, or equal to the pressure of the atmosphere. As soon as *M* begins to descend, the valve *B* closes, and the air between M and B escapes through the valve M. The water will ascend in the pump each time this process is repeated, and will finally pass through the valves B and M; and then, when M ascends to A, it will flow through D.

If h be the altitude of a column of water the pressure of which is equal to that of the atmosphere, BC must always be less than h, otherwise the water would never reach B.

100. If P be the surface of the water in BC, r the radius of the cylinder AB, ρ the density of water, and if we suppose M to ascend very slowly; the pressure of the air in $MP = g\rho(h - PC)$, therefore the pressure upwards on $M = g\rho\pi r^2 (h - PC)$, and the pressure of the atmosphere downwards on $M = g\rho\pi r^2 h$, therefore the tension of the rod $AM = g\rho\pi r^2 . PC$.

101. To find the height through which the water rises each time the piston ascends.

Let P be the surface of the water in BC when M is at B; Q the surface of the water when M is at A. Then, the pressure of the air in $BP = g\rho(h - PC)$, and the pressure of the air in $AQ = g\rho(h - QC)$; (but pressure of the air in BP) : (pressure of the air in AQ) = (vol. AQ) : (vol. BP);

$$\therefore\ h - PC : h - QC = (\text{vol.} AQ) : (\text{vol.} BP).$$

102. When AE is the range of the piston, the pressure of the air between B and M, when M is at E, must be greater than the pressure of the atmosphere, otherwise the air will not escape through the valve in M, and M will reascend without increasing the elevation of the water in BC. Let P be the surface of the water in BC when M is at A. The pressure of the air in the pump $= g\rho(h - PC)$. When M comes to E the pressure of the air in $BE = g\rho(h - PC)\,AB \div EB$; and this must be greater than $g\rho h$, the atmospheric pressure; therefore $AE.h$ must be greater than $AB.PC$. BC is the greatest value of PC, therefore $AE.h$ must be greater than $AB.BC$.

103. Suppose the whole pump to be part of the same cylinder, and the valve to be at, or near the surface of the water. Let AE (fig. 39) be the range of the piston, P the surface of

the water within the pump, C the surface of the water on the outside. When the piston is at A, the pressure of the air in $AP = g\rho(h - PC)$; when the piston descends to E, the pressure of the air in $EP = g\rho(h - PC)AP \div EP$, and this must be greater than $g\rho h$, the atmospheric pressure, in order that the valve in the piston may open, therefore $h.AE$ must be greater than $AP.PC$. The greatest value of $AP.PC$ is $\frac{1}{4}AC^2$, therefore $4h.AE$ must be greater than AC^2.

The Forcing Pump. (Fig. 40.)

104. M is a solid piston working in a hollow cylinder ABC, the lower end of which is immersed in water; DF a tube ascending from AB; B, D valves opening upwards; AE the range of the piston.

Let M be at E, and the pressure of the air in the pump equal to the atmospheric pressure. Let M be elevated to A, then the pressure of the air below M is diminished, and the pressure of the atmosphere will force the water up the tube BC. When M descends the valve B closes, D opens, and a portion of the air between M and B escapes through DF. When M ascends, the water rises in BC as before, and at last rises above B, and is forced up the tube DF when M descends. On elevating M, D closes, a fresh portion of water enters AE through B, and is forced up DF by the next descent of M.

A solid cylinder working in a water-tight collar at A, is frequently used instead of the piston M.

The stream of water may be rendered continuous by means of a close vessel DF (fig. 41) filled with air; H is the lower extremity of the ascending tube. When the surface of the water in DF rises above H, the pressure of the air, which is condensed in the upper part of DF forces the water up HF in a continued stream.

The Fire Engine. (Fig. 42.)

105. AB, $A'B'$, are two forcing pumps, having a common air vessel DF, and suction tube C. The pistons are worked

by a lever LGL', so that one descends while the other ascends. The jet of water may be pointed in any direction by means of the flexible tube F. The action of the engine is in all respects the same as that of the forcing pump.

THE CONDENSER. (Fig. 43.)

106. AB is a hollow cylinder, of which the end B is screwed into the neck of a strong vessel C; M a piston containing a valve opening downwards; B a valve also opening downwards.

Suppose M to be at A, and the barrel AB and the receiver C to be filled with air of the same density as the atmospheric air. When M begins to descend the pressure of the air in MB, which is increased in consequence of the diminution of its volume, closes the valve M, and opens the valve B; and when M is thrust down to B, a quantity of air, which, under the pressure of the atmosphere, occupied the space AB, is forced into C; when M begins to ascend, the pressure of the air in C closes the valve B, and the pressure of the atmosphere opens M, and when M comes to A, AB is filled with air under the pressure of the atmosphere, and this is forced into C by the next descent of M.

To find the density of the air in the receiver after n descents of the piston.

Let A, B, be the capacities of the receiver and barrel respectively; ρ the density of atmospheric air. Then ρA will be the mass of the air contained in the receiver at first, and ρB the mass of the air forced into the receiver at each descent of the piston, therefore $\rho\text{A} + n\rho\text{B}$ will be the mass of the air in the receiver after n descents of the piston; and its volume is A, therefore its density will be

$$\rho\left(1 + n\frac{\text{B}}{\text{A}}\right).$$

107. The gauge of a condenser is a glass tube AB (fig. 44) sealed at A and communicating with the receiver of the condenser at B, the part AP of the tube is filled with air which is

separated from the air in PB by a drop of mercury P. When the air in the receiver is condensed, P is forced towards A, till the pressures, and, therefore, the densities of the air in AP, PB are equal. Let ρ be the density of atmospheric air; then, when the drop of mercury is at M, the density of the air in AM or $MB = \rho \dfrac{AP}{AM}$ (31).

HAWKSBEE'S AIR PUMP. (Fig. 45.)

108. AB, $A'B'$ are two hollow cylinders communicating at B, B', with a strong vessel by means of a pipe C; M, M' pistons containing valves opening upwards, and worked by a toothed wheel E; B, B', valves opening upwards.

Suppose M to be at A, and M' at B', and the density of the air in the receiver C, and in AB, to be equal to the density of atmospheric air. Then if E be turned so that M may descend and M' ascend, the valve B' opens, B and M' close, and a quantity of air, which at first occupied the space AB, is forced through the valve M, by the time M reaches B; when the wheel is turned in the opposite direction, the valve B opens, M and B' close, and a quantity of air, which after the first turn of the wheel occupied the space $A'B'$, is forced through M' by the descent of M' from A' to B'. The exhaustion may be carried on to any required extent, by a repetition of this process.

To find the density of the air in the receiver after any number of turns of the wheel E.

Let A, B, be the capacities of the receiver and barrel respectively; ρ the density of the air in the machine ρ_1, $\rho_2 \dots \rho_n$ the densities of the air after 1, 2,...... n turns. Then, the air, which occupied the space A when M was at B, will occupy the space A + B when M comes to A, therefore $\rho_1 (\text{A} + \text{B}) = \rho \text{A}$, similarly $\rho_2 (\text{A} + \text{B}) = \rho_1 \text{A}$, and so on;

$$\therefore \ \rho_n (\text{A} + \text{B})^n = \rho \text{A}^n.$$

Hence if h be the altitude of the mercury in a barometer, σ the density of mercury, and therefore $g\sigma h$ the pressure of the atmosphere, the pressure of the air in the receiver after n turns will be

$$g\sigma a\left(\frac{A}{A+B}\right)^n.$$

The employment of two pistons worked by the same wheel diminishes considerably the labour of working the pump. For the pressures of the atmosphere on the upper surfaces of M, M' being equal, the pump may be worked by a force sufficient to overcome the friction together with the difference of the pressures on the under surfaces of M, M'; while the ascent of a single piston is opposed by the friction together with the difference between the pressures on its upper and under surfaces.

Smeaton's Air Pump. (Fig. 46.)

109. AB is a hollow cylinder communicating with the receiver by means of the pipe BC; M a piston worked by a rod AM passing through an air-tight collar in a plate which closes the upper end of the cylinder; at A, M, B, are placed valves opening upwards.

Let A, B be the capacities of the receiver and barrel respectively; ρ the density of the air in the machine; and suppose M to be at A. Then, as soon as M begins to descend, the valves A and B will close, and the valve at M will open: when M reascends from B to A, the valves A and B will open, and the valve at M will close; and the air which occupied the space A before M left A, will occupy the space A + B when M returns to A.

Hence, if $\rho_1, \rho_2, \ldots \rho_n$ be the densities of the air in the receiver after 1, 2, ... n descents and ascents of the piston, $\rho_1(A+B) = \rho A$, similarly $\rho_2(A+B) = \rho_1 A$, and so on,

$$\therefore \rho_n(A+B)^n = \rho A^n.$$

The valve A, which closes as soon as M begins to descend, relieves M from the pressure of the atmosphere, and the valve in M is opened by a very small pressure of the air beneath. On this account Smeaton's pump is capable of producing a greater degree of exhaustion than Hawksbee's. Also the removal of the pressure of the atmosphere on M, diminishes the labour of working this pump.

110. The receiver is usually a strong glass jar, having its mouth ground truly plane, placed with its mouth downwards on a plane surface of brass, into which the extremity of the tube C is inserted. The junction of the receiver and the plate of brass is rendered impervious to the air by smearing the edge of the receiver with some unctuous substance. The valves are formed of a triangular piece of oiled silk stretched over a grated orifice in a plate of metal, to which the corners of the triangle are fastened. When the air presses on the upper surface of the valve, the silk is brought into contact with the edge of the orifice, and the passage of the air through it is prevented. When the air presses on the under side of the valve, the silk is lifted up from the grating, and the air finds a free passage between the silk and the plate of the valve.

111. The barometer gauge is a vertical glass tube not less than 31 inches long, the lower end of which is immersed in a cistern of mercury, while its upper end communicates with the receiver.

If z be the altitude of the mercury in the gauge above the surface of the mercury in the cistern, the pressure of the air in the receiver $= g\sigma h - g\sigma z$. Hence, the density of the air in the receiver

$$= \rho \frac{h - z}{h}.$$

112. The syphon gauge is a glass tube $ABCD$ (fig. 47) closed at A, and communicating with the receiver at D; AB and part of BC is filled with mercury. As the exhaustion proceeds, the mercury sinks in AB and rises in BC; and if z be the perpendicular distance between the surfaces of the mercury in AB and BC, $g\sigma z$ will be the pressure at the surface of the mercury in BC, or the pressure of the air in the receiver.

THE COMMON BAROMETER. (Fig. 48.)

113. The principle of this instrument is explained in (28). In order to avoid the trouble of observing the altitudes of both extremities of the column of mercury, the diameter of the tube *BC* is made much greater than that of the tube *AB*, and a scale of inches is attached to *AB*. Let zero of the scale of inches be at *K*; and when the plane of the surface of the mercury in *BC* passes through *K*, let *H* be the upper extremity of the column of mercury. When the extremity of the mercurial column is at *P*, let the plane of the surface of the mercury in *BC* pass through *Q*; and let H, K be the areas of horizontal sections of the tubes *AB*, *BC*, respectively. Then, $\mathrm{H}.HP = \mathrm{K}.KQ$, and *HP* is the apparent variation of the altitude of the mercury; but its real variation

$$= HK - PQ = HP + KQ = \left(1 + \frac{\mathrm{H}}{\mathrm{K}}\right).HP,$$

and the true altitude of the mercury $= PQ = HK - \left(1 + \frac{\mathrm{H}}{\mathrm{K}}\right).HP.$

In some barometers the cistern is constructed as in fig. 49, and the mercury in it is elevated or depressed by a screw, till its surface touches a fine point, which is in the same horizontal plane with the zero of the scale when the tube *AB* is vertical.

In order to obtain the true height of a column of mercury, the pressure of which is equal to that of the atmosphere, we must add the capillary depression of the mercury in *AB* to the observed altitude. The exact amount of the depression in glass tubes of different diameters appears to be rather uncertain. Hence, in order to determine accurately the absolute height of the mercury, the observations must be made with a barometer having a tube of such large internal diameter that the depression may be nearly insensible; or with the syphon barometer (fig. 17) in which, on account of the equality of the tubes *AB*, *BC*, the extremities of the columns of mercury in them are equally depressed.

114. To compare the specific gravities of air (or gas) and water.

Let a large glass flask capable of being closed by a stop-cock, be exhausted as completely as possible, and weighed. Permit the air to enter the flask, and weigh it again. Lastly, weigh the flask when filled with water.

Let X be the weight of the exhausted flask, Y its weight when filled with air, W its weight when filled with water. Then $Y - X$ is the weight of the air contained in the flask, $W - X$ the weight of the water contained in it. Therefore, since $Y - X$, $W - X$, are the weights of equal volumes of air and water respectively,

$$\frac{\text{specific gravity of air}}{\text{specific gravity of water}} = \frac{Y - X}{W - X}.$$

According to the experiments of Biot the specific gravity of water at 20,5° is 768,264 times that of dry atmospheric air at 0°, under the pressure of 29,922 inches of mercury at 0°, in lat. 45°.

115. It has been tacitly assumed in (114) that the exhaustion of the flask can be carried so far that the weight of the air remaining in it may be neglected, and that the temperature and pressure of the atmosphere remain unchanged during the process of weighing. When these conditions cannot be satisfied, let the flask in which the air is weighed be counterpoised by an air-tight flask, the external volume of which is exactly equal to that of the former flask when closed.

Let Y be the apparent weight of the flask filled with air of nearly atmospheric pressure, X its apparent weight when the pressure of the air within it has been diminished as far as possible by exhaustion. Since the weights of the atmospheric air displaced by the two flasks are equal, Y, X will not be affected by any change of density which may take place between the two weighings. z, Π, T; z', Π', T' the weights, pressures, and temperatures of the air contained in it in the former and latter cases respectively; E the expansion of the air for one degree of temperature. Then

$$\frac{z'}{z} = \frac{\Pi'}{\Pi} \frac{1 + ET}{1 + ET'};$$

$$Y - X = Z - Z' = Z\left(1 - \frac{\Pi'}{\Pi}\,\frac{1 + ET}{1 + ET'}\right).$$

Let W be the apparent weight of the flask when filled with water at the temperature S; S, T being nearly equal; V_S, V_T the capacities of the flask at the temperatures S, T; e the cubical expansion of the substance of which the flask is made for one degree of heat. Then $W - X + Z'$ will be the weight of a volume V_S of water at S. $V_T = V_S\{1 - e(S - T)\}$; $\therefore \{1 - e(S - T)\}(W - X + Z')$ will be the weight of a volume V_T of water at S^0. Hence, the specific gravity of water at S^0, divided by the specific gravity of the air at T^0, under the pressure Π,

$$= \{1 - e(S - T)\}\,\frac{W - X + Z'}{Z}$$

$$= \{1 - e(S - T)\}\left(1 - \frac{\Pi'}{\Pi}\,\frac{1 + ET}{1 + ET'}\right)\frac{W - X + \dfrac{\Pi}{\Pi - \Pi'}(Y - X)}{Y - X},$$

very nearly.

When the expansions of the air and of water are accurately known, the ratio of their specific gravities at any other temperatures may be readily computed.

116. To determine the weight of a given volume of water.

Let a sphere, cube, or cylinder, of known dimensions, be weighed in air and in water. Let V be the volume of the sphere; W its apparent weight in air; X its apparent weight when suspended in water; U the weight of the air displaced by it; W′, X′ the weights of the air displaced by the weights W, X. Then, weight of the sphere − weight of the air displaced by it $= W - W'$; weight of the sphere − weight of the water displaced by it $= X - X'$; therefore, the weight of the water displaced by the sphere, or the weight of a volume V of water $= W - X + U - W' + X'$.

117. To compare the specific gravities of a solid and a fluid, by weighing the solid in air and in the fluid.

Let the solid be suspended by a fine wire, or a hair, from the pan of a balance, as in (fig. 50), and let W be the weight of the solid in air, X its apparent weight when suspended in the fluid. Then (22), neglecting the weight of the air displaced by the solid,

weight of the solid − weight of the fluid displaced = X;

therefore, weight of the fluid displaced = W − X;

therefore, since W and W − X are the weights of equal volumes of the solid and of the fluid,

$$\frac{S.G.\text{solid}}{S.G.\text{fluid}} = \frac{W}{W - X}.$$

When great accuracy is required, the weight of the air displaced by the solid must be taken into account.

Let W be the weight of the solid in air, U the weight of the air displaced by it; X the apparent weight of the solid when suspended in water. Then,

weight of the solid − U = W;

weight of the solid − weight of the fluid displaced by it = X;

therefore the weight of the solid = W + U,

and the weight of the fluid displaced by it = W − X + U;

$$\therefore \frac{S.G.\text{solid}}{S.G.\text{fluid}} = \frac{W + U}{W - X + U}.$$

118. When the weight of the solid is less than the weight of the fluid displaced by it, it must be fastened to another solid of sufficient density and magnitude to cause both to sink. Let X = apparent weight of the denser solid in the fluid − apparent weight of both solids in the fluid = weight of the fluid displaced by the rarer solid − weight of the rarer solid;

therefore the weight of the fluid displaced = W + X;

$$\therefore \frac{S.G.\text{solid}}{S.G.\text{fluid}} = \frac{W}{W + X}.$$

If we take into account the weight of the air displaced,

weight of the solid − weight of air displaced by it = W;

weight of the fluid disp^{d}. by the solid − weight of the solid = X;

therefore weight of the fluid displaced = W + U + X;

$$\therefore \frac{S.G.\text{solid}}{S.G.\text{fluid}} = \frac{W + U}{W + U + X}.$$

119. To compare the specific gravities of two fluids by weighing the same solid in each.

Let W be the weight of the solid in air, X its apparent weight when suspended in the fluid (*A*), Y its apparent weight when suspended in the fluid (*B*). Then, neglecting the weight of the air displaced by the solid,

weight of the solid − weight of the fluid (*A*) displaced by it = X;

weight of the solid − weight of the fluid (*B*) displaced by it = Y;

therefore weight of the fluid (*A*) displaced = W − X,

and weight of the fluid (*B*) displaced = W − Y.

W − X, W − Y are the weights of equal volumes of the fluids (*A*) and (*B*) respectively;

$$\therefore \frac{S.G.\text{fluid } (A)}{S.G.\text{fluid } (B)} = \frac{W - X}{W - Y}.$$

If the weight of the air displaced by the solid = U,

weight of the fluid (*A*) displaced = W + U − X;

weight of the fluid (*B*) displaced = W + U − Y;

$$\therefore \frac{S.G.\text{fluid } (A)}{S.G.\text{fluid } (B)} = \frac{W + U - X}{W + U - Y}.$$

120. To compare the specific gravities of two fluids (*A*) and (*B*) by weighing equal volumes of each.

Let X be the weight of a flask filled with the fluid (*A*) and closed with a ground stopper; Y the weight of the flask similarly filled with the fluid (*B*). W the weight of the flask. Then, neglecting the weight of the air contained in the flask,

weight of the fluid (*A*) contained in the flask = X − W;

weight of the fluid (*B*) contained in the flask = Y − W.

X − W, Y − W are the weights of equal volumes of the fluids (*A*) and (*B*) respectively;

$$\therefore \frac{S.G.\text{fluid}\,(A)}{S.G.\text{fluid}\,(B)} = \frac{\text{X} - \text{W}}{\text{Y} - \text{W}}.$$

If the weight of the air contained in the flask = U; then,

weight of the fluid (A) contained in the flask = X − W + U;

weight of the fluid (B) contained in the flask = Y − W + U;

$$\therefore \frac{S.G.\text{fluid}\,(A)}{S.G.\text{fluid}\,(B)} = \frac{\text{X} - \text{W} + \text{U}}{\text{Y} - \text{W} + \text{U}}.$$

121. The specific gravity of a solid broken into small fragments may be found in the following manner.

Let W be the weight of the solid in air; X the weight of a flask filled with the fluid; Y the weight of the flask containing the fragments of the solid, and filled up with the fluid. Then, neglecting the weight of the air displaced by the solid,

weight of the solid − weight of the fluid displaced by it = Y − X;

$\therefore$ weight of the fluid displaced = W − Y + X;

$$\therefore \frac{S.G.\text{ solid}}{S.G.\text{ fluid}} = \frac{\text{W}}{\text{W} - \text{Y} + \text{X}}.$$

If the weight of the air displaced by the solid = U,

$$\frac{S.G.\text{ solid}}{S.G.\text{ fluid}} = \frac{\text{W} + \text{U}}{\text{W} - \text{Y} + \text{X} + \text{U}}.$$

The Common Hydrometer. (Fig. 51.)

122. E, D are two hollow spheres having their centers in the axis of a graduated cylindrical stem EC. D is loaded with lead so that the center of gravity of the whole instrument may be below the center of gravity of the fluid displaced by the spheres E, D. The instrument is used in comparing the specific gravities of fluids.

Let W be the weight of CED, V its volume; K the area of a section of the stem EC. When it floats vertically in a fluid (A), let the surface of the fluid meet EC in P; and when it floats vertically in a fluid (B), let the surface of the fluid meet EC in Q. Then, since V − K.CP, and V − K.CQ

are the volumes of the fluids (A) and (B) displaced by the instrument, and the weight of a floating solid is equal to the weight of the fluid displaced by it (21),

$$\text{W} = \{S.G.\text{fluid}\,(A)\}\,(\text{V} - \text{K}.CP);$$

$$\text{W} = \{S.G.\text{fluid}\,(B)\}\,(\text{V} - \text{K}.CQ);$$

$$\therefore \frac{S.G.\text{fluid}\,(A)}{S.G.\text{fluid}\,(B)} = \frac{\text{V} - \text{K}.CQ}{\text{V} - \text{K}.CP}.$$

Sikes' Hydrometer. (Fig. 52.)

123. This instrument differs from the preceding in the form of the stem EC, which is a very thin flat bar, and in having a series of weights capable of being fixed on the stem connecting E and D, of such magnitude that when DC floats with nearly the whole of its stem above the surface of the fluid, the addition of one of the weights causes it to sink nearly to C.

Let V be the volume of the instrument, W its weight; K the area of a section of the stem EC. When it floats in a fluid (A), let X be the weight at D, P the surface of the fluid; when it floats in a fluid (B), let Y be the weight at D, Q the surface of the fluid; and let R, S be the volumes of the weights X, Y. Then,

the weight of the fluid (A) displaced = W + X;

the weight of the fluid (B) displaced = W + Y;

the volume of the fluid (A) displaced = V + R − K . CP;

the volume of the fluid (B) displaced = V + S − K . CQ;

$$\therefore \text{W} + \text{X} = \{S.G.\text{fluid}\,(A)\}\,(\text{V} + \text{R} - \text{K}.CP);$$

$$\text{W} + \text{Y} = \{S.G.\text{fluid}\,(B)\}\,(\text{V} + \text{S} - \text{K}.CQ);$$

$$\therefore \frac{S.G.\text{fluid}\,(A)}{S.G.\text{fluid}\,(B)} = \frac{(\text{W} + \text{X})\,(\text{V} + \text{S} - \text{K}.CQ)}{(\text{W} + \text{Y})\,(\text{V} + \text{R} - \text{K}.CP)}.$$

Nicholson's Hydrometer. (Fig. 53.)

124. EF is a hollow cylinder; C a dish supported by a slender wire CE placed in the axis of EF; D a heavy dish suspended from the lower extremity of EF. This instrument

is used in comparing either the specific gravity of a fluid with that of a solid, or the specific gravities of two fluids with each other.

(1) To compare the specific gravities of a solid and a fluid.

Let Z be the weight, which placed in C, causes the instrument to sink in the fluid till the surface of the fluid meets EC in a given point H. Place the solid in C and let X be the weight which must be added, to make the instrument sink to H. Place the solid in D, and let Y be the weight which must be placed in C in order to sink the instrument to H. Then, neglecting the weight of the air displaced by the solid,

$$\text{weight of the solid} = Z - X;$$

weight of the solid − weight of the fluid displaced = apparent weight of the solid in the fluid = Z − Y;

$$\therefore \text{ weight of the fluid displaced} = Y - X;$$

$$\therefore \frac{S.G.\,\text{solid}}{S.G.\,\text{fluid}} = \frac{Z - X}{Y - X}.$$

If the weight of the air displaced by the solid = U,

$$\text{weight of the solid} = Z - X + U;$$

$$\therefore \text{ weight of the fluid displaced} = Y - X + U;$$

$$\therefore \frac{S.G.\,\text{solid}}{S.G.\,\text{fluid}} = \frac{Z - X + U}{Y - X + U}.$$

(2) To compare the specific gravities of two fluids (A) and (B).

Let W be the weight of the hydrometer, X the weight which must be placed in C to sink the instrument to H in the fluid (A); Y the weight which must be placed in C to sink the instrument to H in the fluid (B). Then,

$$\text{weight of the fluid } (A) \text{ displaced} = W + X;$$

$$\text{weight of the fluid } (B) \text{ displaced} = W + Y.$$

The volume of the fluid displaced is the same in both cases;

$$\therefore \frac{S.G.\,\text{fluid}\,(A)}{S.G.\,\text{fluid}\,(B)} = \frac{W + X}{W + Y}.$$

MEIKLE'S, OR HARE'S HYDROMETER. (Fig. 54.)

125. *CD*, *EF* are two vertical glass tubes communicating at *E* and *F* with a cavity *G*, which is connected with some contrivance for partially exhausting the air contained in it. *C* and *E* are immersed in cups containing two fluids (*A*) and (*B*), the specific gravities of which are to be compared. If the air in *G* be now partially exhausted the fluids will ascend in the tubes. Let the fluid (*A*) ascend to *P*, and the fluid (*B*) to *Q*; and let the surfaces of the fluids (*A*), (*B*) in the cups meet the tubes *CD*, *EF* in *C*, *E*. Then, if the atmospheric pressure $= \Pi$, and the pressure of the air in $G = \mathrm{M}$, (9)

$$\Pi - \mathrm{M} = \{S.G.\text{fluid }(A)\}.CP;$$

$$\Pi - \mathrm{M} = \{S.G.\text{fluid }(B)\}.EQ;$$

$$\therefore \frac{S.G.\text{fluid }(A)}{S.G.\text{fluid }(B)} = \frac{EQ}{CP}.$$

When the tubes are small, the altitudes *CP*, *EQ* must be diminished by the spaces through which the fluids are elevated by capillary attraction. Or the effect of capillary attraction may be eliminated in the following manner. Let *a*, *b*, be the capillary elevations of the fluids in the tubes *CD*, *EF*; then,

$$\{S.G.\text{fluid }(A)\}.(CP - a) = \{S.G.\text{fluid }(B)\}(EQ - b).$$

Permit some air to enter *G*, and let *P'*, *Q'*, be the extremities of the columns of fluid in *CD*, *EF*; therefore

$$\{S.G.\text{fluid }(A)\}.(CP' - a) = \{S.G.\text{fluid }(B)\}(EQ' - b);$$

$$\therefore \{S.G.\text{fluid }(A)\}\,PP' = \{S.G.\text{fluid }(B)\}\,QQ';$$

$$\therefore \frac{S.G.\text{fluid }(A)}{S.G.\text{fluid }(B)} = \frac{QQ'}{PP'}.$$

SAY'S INSTRUMENT FOR MEASURING THE VOLUMES OF SMALL SOLIDS. (Fig. 55.)

126. *PC* is a glass tube of uniform bore terminating in a cup *PE* having its mouth ground truly plane, and capable of being closed so as to be air-tight by a plate of glass *E*; within *PE* is a cup *B* containing the substance the volume of which is

sought. Take off the plate E, and immerse PC vertically in mercury till the surface of the mercury meets the tube in a given point P; close the cup PE with the plate E, and elevate the tube PC till the surface of the mercury on the outside meets the tube in any point C; and let M be the extremity of the column of mercury within the tube.

Let u be the volume of the space occupied by the air in PE before the solid was placed in the cup B; v the volume of the solid; K the area of a section of the tube PC; h the altitude of the mercury in the barometer; σ the density of mercury. When the surface of the mercury was at P, the air in EP occupied the space $u - v$, and its pressure $= g\sigma h$; when the surface of the mercury within the tube is at M, and the surface of the mercury on the outside at C, the air in EPM occupies the space $u - v + \text{K}.PM$, and its pressure $= g\sigma(h - MC)$; therefore (31)

$$\frac{u - v + \text{K}.PM}{u - v} = \frac{h}{h - MC};$$

$$\therefore\ v = u - \frac{h - MC}{MC}\text{K}.PM.$$

u may be found by a similar process, the cup B being empty. K is found by weighing the mercury occupying a given portion of the tube PC. A cubic inch of mercury at 16° weighs $3429\frac{1}{2}$ grains nearly, therefore if the length of the column of mercury in PC expressed in inches $= a$, and its weight in grains $= w$, $w = 3429\frac{1}{2}$ (vol. mercury in PC) $= 3429\frac{1}{2}.\text{K}a$, K being expressed in square inches.

If the weight of the solid $= \text{W}$, its specific gravity $= \text{W} \div v$. In this manner the specific gravities of powders and soluble substances are found, when the other methods, which require the substances to be immersed in fluid, cannot be used.

THE PIEZOMETER. (Fig. 56.)

127. This instrument, by means of which the compressibility of liquids may be exhibited and measured, consists of a thermometer tube DC open at C, enclosed in a strong glass vessel EF. CD is nearly filled with the liquid to be examined, and then a drop of mercury is introduced to keep the liquid

within the tube separate from the liquid without; *EF* is then filled with water, and the required pressure produced by a piston, which is pressed down by turning a screw *G*. The pressure is measured by means of a gauge *AB* similar to the one described in (107), and the decrement of the volume of the fluid in *CD*, is deduced from the space through which the drop of mercury descends, the area of a section of the tube and the capacity of the bulb *D* having been found by weighing the mercury contained in the bulb and in a given length of the tube.

The apparent diminution of the volume of the fluid in *CD* requires a slight correction for the alteration of the capacity of *D* arising from the compressibility of glass.

The Hydraulic Ram. (Fig. 57.)

128. *AB* is a pipe descending obliquely from a reservoir of water *A*, to an air vessel *G*, into which is inserted the ascending pipe *FH*. *C* is a smaller air vessel; *B* a large valve opening downwards; *D* a valve opening upwards; *E* a small valve opening into *C*.

Suppose the valves *B*, *E*, closed by the pressure of the water in *AB*; *D* closed by its own weight; *G*, *C* filled with air; and *FH* filled with water up to the level of the surface of the water in *A*. Let the valve *B* be depressed; then the water in *AB* will move in the direction *AB*, and flow out at *B*, till the valve *B* carried upwards by the stream closes the orifice at *B*; the water in *AB* having its motion thus suddenly checked, will exert a very great pressure on the inner surface of the pipe *AB*, and will rush into the air vessel *G*, and up the pipe *FH*, compressing at the same time the air in *G* and *C*. As soon as the water in *AB* comes to rest, *D* closes, and the pressure of the air in *C* causes the water in *AB* to recoil till the air in *C* occupies a larger space than it did under the pressure of the atmosphere; at this instant, the pressure at *B* being less than the pressure of the atmosphere, *B* descends, and the action of the machine is renewed.

In this manner the water ascends in *FH* at each successive impulse, till it reaches the place to which it is desired to elevate it. A portion of the air in *G* and *C* is taken up by the water, which absorbs a considerable quantity of air under a high pressure; to supply the waste arising from this cause, the machine is provided with the valve *E*, which opens and permits the air to enter, during the recoil of the water in *AB*.

The Atmospheric Steam Engine. (Fig. 58.)

129. *AB* is a hollow cylinder communicating with a boiler by means of a pipe *C*; *B* a valve opening downwards and closed by a spring; *ED* a pipe leading from a cistern of cold water *E*; *M* a piston connected with one extremity of a lever *LGF*; from the other extremity of the lever is suspended *FH*, the rod by which the machinery worked by the steam-engine is put in motion. *H* is a weight equal to half the pressure of the atmosphere on the upper surface of *M*. An apparatus connected with *FL* opens the cock *C*, when *M* descends to *B*, and closes it when *M* ascends to *A*. The cock *D* is opened in the same manner when *M* comes to *A*, and is closed again soon after *M* begins to descend.

Suppose *M* to be at *B*, and the pressure of the steam in the boiler a little greater than the pressure of the atmosphere; then, when *C* is opened, the steam rushes into *MB*, and the pressures upon the upper and lower surfaces of *M* being nearly equal, the weight *H* will cause *M* to ascend. When *M* comes to *A*, *C* is closed, and *D* is opened; a jet of cold water issues into the cylinder and condenses the steam, leaving a vacuum below *M*; and since the pressure of the atmosphere on *M* is equal to twice the weight of *H*, *M* will descend with a moving force equal to the weight of *H*. When *M* arrives at *B*, *C* is opened again, and *M* ascends as before.

The water remaining in *MB* escapes through the valve *B*, which is forced open by the pressure of the steam when first admitted.

Watt's Steam Engine. (Fig. 59.)

130. *AB* is a hollow cylinder closed at both ends. *LGF* is a lever, one end of which is connected with the piston *M*, by a rod *AM* passing through a steam tight collar at *A*; the

other end of the lever is attached to the crank of a fly wheel. *D* is a vessel, called the condenser, surrounded by cold water; *RS* a tube connecting *AB* with the boiler and with *D*. At *R* and *S* are placed valves connected with the fly wheel in such a manner that when *M* comes to *A*, a communication is opened between *AM* and the boiler, and between *MB* and the condenser, and is closed again when *M* has described one third of *AB*; and when *M* comes to *B*, a communication is opened between *MB* and the boiler, and *AM* and the condenser, and closed, as in the former case, when *M* has described one third of *BA*.

Suppose *M* to ascend from *B* to *A*, the space below *M* being filled with steam from the boiler; as soon as *M* arrives at *A*, the communication is opened between *MB* and *D*, and the steam in *MB* flows into *D*, and is there condensed leaving a vacuum in *MB*; at the same time, a communication being open between *AM* and the boiler, steam flows into *AM*, and *M* is forced downwards by the full pressure of the steam, during one third of its descent, and after the communication between *AM* and the boiler is cut off, by the diminished pressure of the steam in the cylinder. In the same manner, when *M* arrives at *B*, a vacuum is produced in *AM* by the condensation of the steam, and *M* is pressed upwards by the steam admitted below.

The condensation of the steam in *D* is promoted by a jet of cold water, which is removed as fast as it collects by a pump *P*.

A description of the contrivances for regulating the supply of steam and water, and for making the extremity of the piston rod describe a curve approaching to a straight line; as well the enumeration of the advantages of this construction over the atmospheric engine, would be improper in this place, on account of its length.

THE HIGH PRESSURE STEAM ENGINE.

131. The construction of the cylinder, piston, and valves in this engine is the same as in Watt's engine. The steam has a pressure many times greater than the pressure of the atmo-

sphere, and instead of being condensed after each stroke of the piston, it is permitted to escape into the open air.

Suppose M to ascend from B to A, the space MB being filled with steam from the boiler; as soon as M arrives at A, a communication is opened between MB and the air, steam from the boiler flows into AM, and M is forced towards B by the excess of the pressure of the steam above the atmospheric pressure. In the same manner when M arrives at B, a communication is opened between AM and the atmosphere, and M is forced towards A by the excess of the pressure of the steam admitted into MB above the atmospheric pressure.

APPENDIX.

Art. 132. The propositions which form the subject of (3), (4), are sometimes illustrated by a feigned experiment, in the following manner.

Let a vessel $ABCD$ (fig. 60) closed on all sides, and exactly filled with fluid, be placed with the side AD horizontal, and therefore free from any pressure arising from the weight of the fluid. At E and F make two equal orifices, and apply any pressure to the surface of the fluid at E by means of a piston; then, in order to prevent the fluid from escaping at F, another piston must be applied and pressed with exactly the same force as that at E. Thus the pressure communicated perpendicularly downwards at E, has, by the intervention of the fluid, been made to act with the same force perpendicularly upwards at F. If the orifice F, instead of being made in the upper horizontal surface of the vessel, be made at any point G in the inclined side, and a force applied sufficient to counteract the effort made by the fluid to escape; then, if any pressure be applied at E, it will be found, as before, that an additional pressure, equal to that at E, must be applied at G perpendicular to the side CD, to preserve the equilibrium. We may conclude therefore that a force impressed on a given surface in any part of a fluid, produces an equal pressure on an equal surface in any other part of the fluid.

133. To find the pressure of a fluid on any surface.

Suppose the surface (S) divided into an indefinite number of portions A, B, C, &c. so small that every point of any one of them may be considered as at the same perpendicular depth below the surface of the fluid; and let their respective perpendicular depths be a, b, c, &c.; then (9) the pressure of the fluid on any one of them $A = g\rho A a$, ρ being the density of the fluid; similarly, the pressure on $B = g\rho Bb$, &c.; therefore the sum of the pressures

$$= g\rho (Aa + Bb + Cc + \&c.).$$

But if the depth of the center of gravity of S below the surface of the fluid $= X$; then, since A, B, C, &c. may be considered as bodies the perpendicular distances of which from the surface of the fluid are a, b, c, &c., (Snowball's Mechanics, 154.)

$$Aa + Bb + Cc + \&c. = (A + B + C + \&c.)\, X = SX;$$

$$\therefore \text{ the pressure of the fluid on } S = g\rho SX.$$

134. A vessel of the form of a cone with its base downwards, is filled with fluid; to compare the pressure on the base of the vessel with the weight of the fluid contained in it.

Let the cone be generated by the revolution of the right-angled triangle ABC (fig. 61) round AC, ρ the density of the fluid; then, the area of the base of the cone $= \pi . CB^2$, and the depth of its center of gravity below the surface of the fluid $= AC$; therefore the pressure on the base of the cone $= g\rho\pi . AC . BC^2$; and the weight of the fluid in the cone $= \frac{1}{3} g\rho\pi AC . BC^2$; therefore pressure on the base of the cone $= 3 .$ (weight of the fluid).

Since the pressure on the base of a vessel filled with a given fluid, depends only on its area, and the depth of its center of gravity below the surface of the fluid, the pressure on the base of the cone BAB' (fig. 61) is equal to the pressure on the base of the cylinder $BAA'B'$ (fig. 62), or the pressure on the base of the truncated cone $BAA'B'$ (fig. 63); the area of the base and the depth of its center of gravity below the surface of the fluid being the same in each case.

135. A hollow sphere is just filled with fluid; to compare the pressure on the internal surface of the sphere with the weight of the fluid.

Let a be the radius of the sphere; then, the area of the surface of the sphere $= 4\pi a^2$, and the depth of its center of gravity below the surface of the fluid $= a$, therefore the pressure on the surface of the sphere $= g\rho 4\pi a^3$, and the weight of the fluid contained in the sphere $= g\rho\frac{4}{3}\pi a^3$; therefore the pressure on the surface of the sphere $= 3$ (weight of the fluid.)

136. To find the center of pressure of the triangle AOB (fig. 64) having the side OA perpendicular to the surface of the fluid, and the side OB in the surface.

Draw HR parallel to BO; and let X, Y be the distances of the center of pressure from OB, OA respectively; $OH = x$. Then, (14)

$$X\int_x\int_y x = \int_x\int_y x^2, \quad Y\int_x\int_y x = \int_x\int_y xy;$$

the integrals being taken between the limits corresponding to the boundary of the figure.

$$\int_y x = xy + C; \quad \int_y^{0,\,HR} x = x.HR = OB\left(x - \frac{x^2}{OA}\right);$$

$$\int_x\left(x - \frac{x^2}{OA}\right) = \frac{x^2}{2} - \frac{x^3}{3OA} + C;$$

$$\int_x^{0,\,OA}\left(x - \frac{x^2}{OA}\right) = \frac{OA^2}{2} - \frac{OA^3}{3OA} = \tfrac{1}{6}OA^2;$$

$\therefore \int_x\int_y x$, between the proper limits, $= \frac{1}{6}OB.OA^2$.

$$\int_y x^2 = x^2y + C; \quad \int_y^{0,\,HR} x^2 = x^2.HR = OB.\left(x^2 - \frac{x^3}{OA}\right);$$

$$\int_x\left(x^2 - \frac{x^3}{OA}\right) = \frac{x^3}{3} - \frac{x^4}{4OA} + C;$$

$$\int_x^{0,\,OA}\left(x^2 - \frac{x^3}{OA}\right) = \frac{OA^3}{3} - \frac{OA^4}{4OA} = \tfrac{1}{12}OA^3;$$

$\therefore \int_x\int_y x^2$, between the proper limits, $= \frac{1}{12}OB.OA^3$.

$$\int_y xy = \tfrac{1}{2}xy^2 + C;$$

$$\int_y^{0,\,HR} xy = \tfrac{1}{2}x.HR^2 = \tfrac{1}{2}BO^2.\left(x - \frac{2x^2}{OA} + \frac{x^3}{OA^2}\right);$$

$$\int_x\left(x - \frac{2x^2}{OA} - \frac{x^3}{OA^2}\right) = \frac{x^2}{2} - \frac{2x^3}{3OA} + \frac{x^4}{4OA^2} + C;$$

$$\int_x^{0,\,OA}\left(x - \frac{2x^2}{OA} + \frac{x^3}{OA^2}\right) = \frac{OA^2}{2} - \frac{2OA^3}{3OA} + \frac{OA^4}{4OA^2} = \tfrac{1}{12}OA^2;$$

$\therefore \int_x \int_y xy$, between the proper limits, $= \frac{1}{8} OA^2 OB^2$;

$$\therefore X = \tfrac{1}{2} OA, \quad Y = \tfrac{1}{2} OB.$$

137. To find the center of pressure of a semicircle *ORA* (fig. 65) having its diameter *OA* perpendicular to the surface of the fluid, and the extremity *O* of the diameter in the surface of the fluid.

Let the plane of the semicircle meet the surface of the fluid in Oy. Draw HR parallel to Oy; and let $OA = 2a$, $OH = x$.

$$\int_y x = xy + C; \quad \int_y^{0,\,HR} x = x.HR = x\sqrt{(2ax - x^2)};$$

$$\int_x^{0,\,2a} x\sqrt{(2ax - x^2)} = \frac{\pi}{2}.a^3;$$

$\therefore \int_x \int_y x$, between the proper limits, $= \frac{\pi}{2} a^3$.

$$\int_y x^2 = x^2 y + C; \quad \int_y^{0,\,HR} x^2 = x^2.HR = x^2\sqrt{(2ax - x^2)};$$

$$\int_x x^2 \sqrt{(2ax - x^2)} = C - \tfrac{1}{4} x (2ax - x^2)^{\frac{3}{2}}$$
$$- \tfrac{5}{4}.\tfrac{1}{3} a (2ax - x^2)^{\frac{3}{2}} + \tfrac{5}{4} a^2 \int_x \sqrt{2ax - x^2};$$
$$\int_x^{0,\,2a} x^2 \sqrt{(2ax - x^2)} = \tfrac{5}{8} \pi a^4;$$

$\therefore \int_x \int_y x^2$, between the proper limits, $= \frac{5}{8} \pi a^4$.

$$\int_y xy = \tfrac{1}{2} xy^2 + C; \quad \int_y^{0,\,HR} xy = \tfrac{1}{2} x HR^2 = \tfrac{1}{2} x (2ax - x^2);$$

$$\int_x (2ax^2 - x^3) = \tfrac{2}{3} a x^3 - \tfrac{1}{4} x^4 + C; \quad \int_x^{0,\,2a} (2ax^2 - x^3) = \tfrac{4}{3}.a^4;$$

$\therefore \int_x \int_y xy$, between the proper limits, $= \frac{2}{3} a^4$.

$$\text{Therefore } X = \frac{5}{4} a, \quad Y = \frac{4}{3}.\frac{a}{\pi}.$$

138. To find the center of pressure of the sector *AOB* (fig. 66) having its center *O* in the surface of the fluid, and the radius *OA* perpendicular to the surface.

Let the plane of the sector meet the surface of the fluid in Oy. Draw the radii OR, OS; and with the center O describe the arcs PP', QQ'. Let the density of the fluid $= \rho$, $OA = a$, $AOB = \alpha$, $OP = r$, $PQ' = \delta r$, $AOR = \theta$, $ROS = \delta\theta$.

Then

d_r (press. on $PP'O$) δr = press. on $PQ = g\rho r^2 \cos\theta . \delta\theta . \delta r$ ult.;

$$\text{press. on } ROS = g\rho \cos\theta . \delta\theta \int_r^{0\,a} r^2 = g\rho \cos\theta . \delta\theta . \frac{a^3}{3};$$

d_θ(press. on AOR) $\delta\theta$ = press. on $ROS = \frac{1}{3} g\rho a^3 \cos\theta . \delta\theta$ ult.;

$$\text{press. on } AOB = \tfrac{1}{3} g\rho a^3 \int_\theta^{0\,\alpha} \cos\theta = \tfrac{1}{3} g\rho a^3 \sin\alpha.$$

d_r (mom. press. on $PP'O$ round Oy) δr = mom. press. on PQ round $Oy = g\rho r^3 (\cos\theta)^2 . \delta r . \delta\theta$ ultimately;

$$\text{mom. press. on } ROS \text{ round } Oy = g\rho (\cos\theta)^2 . \delta\theta \int_r^{0\,a} r^3$$

$$= g\rho (\cos\theta)^2 . \delta\theta \frac{a^4}{4};$$

d_θ (mom. press. on AOR round Oy) $.\delta\theta$

= mom. press. on ROS round $Oy = \frac{1}{4} g\rho a^4 (\cos\theta)^2 . \delta\theta$ ult.;

mom. press. on AOB round Oy

$$= \tfrac{1}{4} g\rho a^4 \int_\theta^{0\,\alpha} (\cos\theta)^2 = \tfrac{1}{16} g\rho a^4 (2\alpha + \sin 2\alpha).$$

d_r (mom. press. on $PP'O$ round OA) $.\delta r$

= mom. press. on PQ round $OA = g\rho r^3 \sin\theta \cos\theta . \delta r . \delta\theta$ ult.;

mom. press. on ROS round OA

$$= g\rho \sin\theta \cos\theta . \delta\theta \int_r^{0\,a} r^3 = g\rho \sin\theta . \cos\theta . \delta\theta \tfrac{1}{4} a^4;$$

d_θ (mom. press. on AOR round OA) $\delta\theta$

= mom. press. on ROS round $OA = \frac{1}{4} g\rho a^4 \sin\theta \cos\theta . \delta\theta$ ult.;

mom. press. on AOB round OA

$$= \tfrac{1}{4} g\rho a^4 \int_\theta^{0\,\alpha} \sin\theta \cos\theta = \tfrac{1}{8} g\rho a^4 (\sin\alpha)^2;$$

$$\therefore\ X.\tfrac{1}{3}g\rho a^3\sin\alpha = \tfrac{1}{16}g\rho a^4(2\alpha+\sin 2\alpha),$$

$$Y.\tfrac{1}{3}g\rho a^3\sin\alpha = \tfrac{1}{8}g\rho a^4(\sin\alpha)^2;$$

$$\therefore\ X = \frac{3}{8}a\left(\frac{\alpha}{\sin\alpha}+\cos\alpha\right),\quad Y = \frac{3}{8}a\sin\alpha.$$

139. A hemispherical bell is placed with its mouth downwards on a horizontal plane, and water is poured into the bell through a hole in its vertex; to find how high the water will rise without lifting the bell.

Let BAB (fig. 67) be a section of the bell made by a plane through its axis AC, BCB a section of the horizontal plane, PHP' a section of the surface of the water. Draw BQ parallel to AC meeting HP in Q; and let ρ be the density of the water, W the weight of the bell. The pressure of the water on the interior of the bell, estimated vertically upwards, is equal to the weight of the superincumbent column of fluid, or the weight of a quantity of fluid of the same bulk as the solid generated by the revolution of BPQ round $AC = \frac{1}{3}\pi g\rho HC^3$; and when $\frac{1}{3}\pi g\rho HC^3 = W$, the weight of the bell is sustained by the pressure of the water.

140. A hollow sphere just filled with fluid, is divided into two parts by a vertical plane through its center; the two hemispheres are held together by ligaments at their highest and lowest points; to find the tensions of the ligaments.

Let the circle APQ (fig. 68) be the section of the sphere made by the vertical plane; P, Q, the highest and lowest points in the circle APQ, C its center, K its center of pressure, G the center of gravity of one of the hemispheres. Then, (17) the pressure on each hemisphere resolved in a direction perpendicular to APQ, is equal to the pressure on the circle APQ; and it acts in a line passing through K; and the resultant of the vertical pressure of the fluid on either hemisphere will not be altered if we suppose the fluid in the hemisphere to become solid; it will therefore be equal to the weight of the fluid in the hemisphere, and will act downwards in a vertical through G. Hence if P, Q, be the tensions of the ligaments at P, Q respectively, $P + Q =$ pressure on APQ; and $P.PQ =$ (pressure on APQ).KQ + (weight of fluid in hemisphere).CG.

If the radius of the sphere $= a$, and the density of the fluid $= \rho$, the pressure on $APQ = g\rho\pi a^3$, weight of fluid in hemisphere $= g\rho\frac{2}{3}\pi a^3$, $CG = \frac{3}{8}a$, and $PK = \frac{5}{4}a$; $\therefore KQ = \frac{3}{4}a$;

$$\therefore P + Q = g\rho\pi a^3, \quad P.2a = g\rho\pi a^3.\tfrac{3}{4}a + g\rho\tfrac{2}{3}\pi a^3.\tfrac{3}{8}a;$$

$$\therefore P = \tfrac{1}{2}g\rho\pi a^3, \quad Q = \tfrac{1}{2}g\rho\pi a^3.$$

141. A rod AB (fig. 69) of uniform thickness, suspended by a string EL, rests with one end immersed in a fluid; to find AP the portion of the rod immersed, and the tension of the string.

Let κ be the area of a section of the rod, W its weight, G its center of gravity, ρ the density of the fluid. Bisect AP in F; and through F, G draw FM, GN, vertical. The resultant of the pressure of the fluid on the rod $=$ weight of the fluid displaced $= g\rho\kappa.AP$; and it acts in the line FM; the other forces are W acting in GN, and T in EL. Therefore, (22)

$$T + g\rho\kappa.AP = W; \quad g\rho\kappa.AP.FE = W.GE;$$

$$\therefore AP^2 - 2AE.AP + 2\frac{W.GE}{g\rho\kappa} = 0;$$

from this equation AP and therefore T may be found.

142. A ship sailing out of the sea into a river, sinks through the space b; on throwing overboard a weight P the ship rises through the space c; to find the weight of the ship.

Let ρ, σ be the densities of fresh and salt water respectively, A the area of the plane of floatation of the ship, W the weight of the ship, V the volume of the salt water displaced by it; then, (21) $W = g\sigma V$, and the volume of the fresh water displaced at first $= V + bA$; $\therefore W = g\rho(V + bA)$; and the volume of the fresh water displaced after the weight P is thrown overboard $= V + (b - c)A$; $\therefore W - P = g\rho\{V + (b - c)A\}$.

Eliminating A, V, we obtain $\left(1 - \frac{\rho}{\sigma}\right) W = \frac{b}{c}P$.

143. A triangular prism floats with its axis horizontal, and one edge immersed; to find its positions of equilibrium.

Let RSD, AB (fig. 70) be sections of the prism and of the plane of floatation, made by a plane perpendicular to the axis of the prism, passing through G its center of gravity. Let W be the weight of the prism, h the length of its axis, ρ the density of the fluid. Draw GE perpendicular to RD, GF perpendicular to SD. Take $PD = \frac{2}{3}AD$, $QD = \frac{2}{3}BD$; and bisect PQ in H. Then H is the center of gravity of the fluid displaced; therefore (21) GH is perpendicular to AB or PQ; and $\frac{1}{2}g\rho h.AD.BD.\sin D = W$.

$$PG^2 = DG^2 + PD^2 - 2ED.PD,$$

$$QG^2 = DG^2 + QD^2 - 2FD.QD;$$

and $PG = QG$, for $PH = QH$, and GH is perpendicular to PQ;

$$\therefore\ PD^2 - QD^2 - 2ED.PD + 2FD.QD = 0;$$

$$\therefore\ AD^2 - BD^2 - 3ED.AD + 3FD.BD = 0;$$

$$\text{and } BD = \frac{2W}{g\rho h \sin D.AD};$$

$$\therefore\ AD^4 - 3ED.AD^3 + \frac{6FD.W}{g\rho h \sin D}AD - \frac{4W^2}{g^2\rho^2h^2(\sin D)^2} = 0.$$

The last term of this equation is negative, and therefore one root of it is negative; but the nature of the question excludes all negative values of AD and BD. Hence, there cannot be more than three positions of equilibrium as long as the same edge is immersed. All values of AD greater than RD, and of BD greater than SD are likewise inadmissible.

144. Two equal rods RD, SD, (fig. 71) meeting each other at right angles, float with the angle D immersed; to find their positions of equilibrium.

Let G be the center of gravity of the rods; P, Q, the middle points of the portions immersed; GR perpendicular to DR; GS perpendicular to DS; GH perpendicular to PQ; HN perpendicular to RD; $2c$ the sum of the lengths of the immersed portions, when the weight of the fluid displaced is equal to the weight of the rods; $RD = e$; $PD = a$; $QD = b$. Then, $a + b = c$; and the center of gravity of the fluid displaced must be in GH, it must also be in PQ, therefore H is the center of gravity of the fluid displaced;

$$\therefore \frac{b}{a} = \frac{PH}{QH} = \frac{PN}{DN}; \quad \therefore \frac{a+b}{a} = \frac{PD}{ND}; \quad \therefore ND = \frac{a^2}{c}.$$

The equation to PQ referred to the axes DR, DS, is $\frac{y}{b} + \frac{x}{a} = 1$; the co-ordinates of G are e, e, therefore the equation to GH is $b(y-e) = a(x-e)$; and H is the intersection of GH and PQ;

$\therefore (b^2+a^2)\,DN = ab^2 + (a^2-ab)\,e$; $\therefore (b^2+a^2)\,a = cb^2 + (a-b)\,ec$;

$$\therefore \{(c-a)^2 + a^2\}\,a = c\,(c-a)^2 + (2a-c)\,ec;$$

$$\therefore 2a^3 - 3ca^2 + (3c-2e)\,ac - (c-e)\,c^2 = 0,$$

one root of this equation is $\frac{1}{2}c$, the other two are

$$\tfrac{1}{2}\{c + \sqrt{(4\tfrac{e}{c} - 3)}\}, \quad \tfrac{1}{2}\{c - \sqrt{(4\tfrac{e}{c} - 3)}\}.$$

145. To find (M) the metacentre of the prism RDS (fig. 70), the prism being inclined in the plane RDS.

The moment of inertia of the plane of floatation round an axis through its center of gravity perpendicular to $RDS = \frac{1}{3}.\frac{1}{4}AB^2.AB.h$; and the volume of the fluid displaced $= \frac{1}{2}h.AD.DB.\sin D$; $\therefore$ (26) $\frac{1}{2}.h.AD.DB.\sin D.HM = \frac{1}{12}AB^3.h$;

$$\therefore HM = \frac{1}{6}\frac{AB^3}{AD.DB\sin D}.$$

146. To find the metacentre of a cone floating with its axis vertical.

Let DC (fig. 72) be the axis of the cone, meeting the plane of floatation in C, CA the radius of the plane of floatation, H the center of gravity of the fluid displaced. Then $DH = \frac{3}{4}DC$; the moment of inertia of the plane of floatation round a horizontal axis through its center of gravity $C = \frac{1}{4}\pi AC^4$; and the volume of the fluid displaced $= \frac{1}{3}\pi AC^2.DC$; $\therefore \frac{1}{3}\pi AC^2.DC.HM = \frac{1}{4}\pi AC^4$;

$$\therefore HM = \frac{3}{4}\frac{AC^2}{DC}; \quad DM = \frac{3}{4}\frac{DC^2 + AC^2}{DC} = \frac{3}{4}\frac{AD^2}{DC}.$$

147. A conical vessel partly filled with fluid, floats in the same fluid with its axis vertical; to find whether the equilibrium of the vessel is stable or unstable.

Let DM (fig. 78) be the axis of the cone making a very small angle with the vertical; C, c the points in which it cuts the plane of floatation and the surface of the fluid within; H, h the centers of gravity of the fluid displaced and of the fluid contained in the cone, when the axis of the cone was vertical; M, m the points in which verticals through the centers of gravity of the fluid displaced and of the fluid within ultimately intersect DM; G the center of gravity of the cone. The weight of the fluid displaced, the weight of the cone, and the weight of the fluid contained in the cone, act in parallel lines through M, G, m, respectively. And the pressures of the exterior and interior fluids will tend to diminish or increase the inclination of DC according as (weight of fluid bDa).mG is greater or less than (weight of fluid displaced).MG.

$$MD = \frac{3}{4}\frac{AD^2}{CD}, \quad mD = \frac{3}{4}\frac{aD^2}{cD},$$

weight of fluid bDa : weight of fluid displaced $= cD^3 : CD^3$; therefore the equilibrium of the cone will be stable or unstable according as $cD^2(GD.cD - \frac{3}{4}aD^2)$ is greater or less than $CD^2(\frac{3}{4}AD^2 - GD.CD)$.

148. An open vessel containing fluid is made to revolve round a vertical axis with the angular velocity a; to find the form of the surface of the fluid.

Let the axis of revolution be made the axis of z; and let z be measured downwards. The forces on the fluid at any point P whose co-ordinates are x, y, z, are, g acting downwards, and $a^2\sqrt{(x^2+y^2)}$ acting in the direction of a perpendicular from P on the axis of revolution; this force may be resolved into

$$a^2\sqrt{(x^2+y^2)}\frac{x}{\sqrt{(x^2+y^2)}}, \text{ or } a^2x, \text{ in a direction parallel to}$$

the axis of x, and a^2y, in a direction parallel to the axis of y. We have then, $X = a^2x$, $Y = a^2y$, $Z = g$; $\therefore d_xp = \rho a^2x$, $d_yp = \rho a^2 y$, $d_zp = \rho g$;

$$\therefore p = \rho\left\{\tfrac{1}{2}a^2(x^2+y^2) + gz\right\} + C.$$

Let the surface of the fluid cut the axis of z at the depth c below the origin; and let Π be the pressure of the atmosphere. Then, $\Pi = \rho g c + C$; $\therefore\ p - \Pi = \rho\left\{\frac{1}{2}a^2(x^2+y^2) + g(z-c)\right\}$.

At any point in the surface of the fluid $p = \Pi$, therefore the equation to the surface of the fluid is

$$0 = a^2(x^2+y^2) + 2g(z-c);$$

the equation to a paraboloid generated by the revolution of a parabola the latus rectum of which $= 2g \div a^2$.

149. A hollow parallelopiped OF (fig. 74) just filled with fluid, revolves round the edge OC, which is vertical, with the angular velocity a; to find the pressure on the side BF, and the center of pressure of the side BF.

Let $OA = a$, $OB = b$, $OC = c$; OA, OB, OC, the axes of x, y, z, respectively; p the pressure at the point (x, y, z). Then, $p = \rho\left\{\frac{1}{2}a^2(x^2+y^2) + gz\right\} + C$; at O $p = 0$, $x = 0$, $y = 0$, $z = 0$; $\therefore\ C = 0$;

$$\therefore\ p = \rho\left\{\tfrac{1}{2}a^2(x^2+y^2) + gz\right\}.$$

Draw HPR, KQS, parallel to BD; MP, NQ parallel to BE; and let $BM = x$, $BH = z$.

The pressure at $P = \rho\left\{\frac{1}{2}(x^2+b^2) + gz\right\}$;

the pressure on $PQ = \rho\left\{\frac{1}{2}(x^2+b^2) + gz\right\} MN.HK$ ult.

the pressure on $KP = \rho\left\{\frac{1}{2}(\frac{1}{3}x^3 + b^2x) + gzx\right\}.HK$ ult.

the pressure on $KR = \rho\left\{\frac{1}{2}(\frac{1}{3}a^3 + b^2a) + gza\right\}.HK$ ult.

the pressure on $BR = \rho\left\{\frac{1}{2}(\frac{1}{3}a^3 + b^2a)z + \frac{1}{2}gz^2a\right\}$;

the pressure on $BF = \rho\left\{\frac{1}{2}(\frac{1}{3}a^3 + b^2a)c + \frac{1}{2}gc^2a\right\}$.

mom. press. on PQ rd. $BE = \rho\left\{\frac{1}{2}a^2(x^2+b^2) + gz\right\}x.MN.HK$ ult.

mom. press. on $KP = \rho\left\{\frac{1}{2}a^2(\frac{1}{4}x^4 + \frac{1}{2}b^2x^2) + \frac{1}{2}gzx^2\right\}.HK$ ult.

mom. press. on $KR = \rho\left\{\frac{1}{2}a^2(\frac{1}{4}a^4 + \frac{1}{2}b^2a^2) + \frac{1}{2}gza^2\right\}.HK$ ult.

mom. press. on $BR = \rho\left\{\frac{1}{2}a^2(\frac{1}{4}a^4 + \frac{1}{2}b^2a^2)z + \frac{1}{4}gz^2a^2\right\}$;

mom. press. on BF rd. $BE = \rho\left\{\frac{1}{2}a^2(\frac{1}{4}a^4 + \frac{1}{2}b^2a^2)c + \frac{1}{4}gc^2a^2\right\}$.

mom. press. on PQ rd. $BD = \rho\left\{\frac{1}{2}a^2(x^2+b^2) + gz\right\}z.MN.HK$ ult.

mom. press. on $KP = \rho\left\{\frac{1}{2}a^2(\frac{1}{3}x^3 + b^2x) + gzx\right\}z.HK$ ult.

mom. press. on $KR = \rho\{\frac{1}{2}a^2(\frac{1}{3}a^3 + b^2a) + gza\}z.HK$ ult.

mom. press. on $BR = \rho\{\frac{1}{4}a^2(\frac{1}{3}a^3 + b^2a)z^2 + \frac{1}{3}gz^3a\}$;

mom. press. on BF rd. $BD = \rho\{\frac{1}{4}a^2(\frac{1}{3}a^3 + b^2a)c^2 + \frac{1}{3}gc^3a\}$.

Hence if X, Z, be the co-ordinates of the center of pressure of BF, referred to the axes BD, BE,

$$X\{\tfrac{1}{2}a^2(\tfrac{1}{3}a^2 + b^2) + \tfrac{1}{2}gc\} = \tfrac{1}{2}a^2(\tfrac{1}{4}a^3 + \tfrac{1}{2}b^2a) + \tfrac{1}{4}gca.$$

$$Z\{\tfrac{1}{2}a^2(\tfrac{1}{3}a^2 + b^2) + \tfrac{1}{2}gc\} = \tfrac{1}{4}a^2(\tfrac{1}{3}a^2 + b^2)c + \tfrac{1}{3}gc^2.$$

150. A solid floating in equilibrium is slightly elevated or depressed, and then left to itself; to determine its motion, without taking into account the resistance of the fluid.

Let ADB (fig. 14) be the position of the solid at the time t, CP a vertical meeting the surface of the fluid aCb in C and the plane of floatation APB in P, $PC = x$, a the space through which the solid was elevated or depressed, A the area of the plane of floatation, V the volume of the fluid displaced by the solid when at rest, ρ the density of the fluid. The moving force on the solid in the direction PC will be the difference between its weight and the weight of the fluid displaced $= g\rho A.CP$, and the mass of the solid $= \rho V$, therefore the accelerating force, tending to diminish x, on the solid in the direction $PC = g\frac{A}{V}.CP$;

$$d_t^2x = -g\frac{A}{V}x, \quad 2d_tx\,d_t^2x = -g\frac{A}{V}2x\,d_tx, \quad (d_tx)^2 = C - g\frac{A}{V}x^2;$$

$$d_tx = 0 \text{ when } x = a; \quad \therefore 0 = C - g\frac{A}{V}a^2; \quad \therefore (d_tx)^2 = g\frac{A}{V}(a^2 - x^2).$$

If we reckon t from the time when $x = a$,

$$d_xt = \frac{1}{d_tx} = -\sqrt{\left(\frac{V}{gA}\right)}\frac{1}{\sqrt{(a^2 - x^2)}};$$

$$\therefore t = \sqrt{\left(\frac{V}{gA}\right)}\cos^{-1}\frac{x}{a}.$$

When $x = 0$, $t = \sqrt{\left(\frac{V}{gA}\right)}\frac{\pi}{2}$, hence the time of an oscillation

$$= \pi\sqrt{\left(\frac{V}{gA}\right)}.$$

151. To determine the small oscillations of the solid DC (fig. 15) after its equilibrium has been slightly disturbed, as in (25), the solid being symmetrical with respect to the plane ADB.

Let the figure represent the position of the solid at the time t; and let ρ be the density of the fluid, K the radius of gyration of the solid revolving round G in the plane ADB, $HG = c$; then, retaining the notation of (26) the moment of the pressure of the fluid, tending to turn the solid round G in the direction FMG,

$$= g\rho V . GM . \theta = g\rho V . \left(k^2 \frac{A}{V} - c\right) . \theta.$$

This pressure tends to diminish θ, and the moment of inertia of the solid round G in the plane $ADB = K^2 \rho V$;

$$\therefore\ d_t^2 \theta = - \frac{g}{K^2} . \left(k^2 \frac{A}{V} - c\right) . \theta.$$

Hence, as in the preceding problem, if $\theta = a$, when $d_t \theta = 0$,

$$t = \frac{K}{\sqrt{\left\{g\left(k^2 \frac{A}{V} - c\right)\right\}}} \cos^{-1} \frac{\theta}{a};$$

and the time of an oscillation $= \pi \dfrac{K}{\sqrt{\left\{g\left(k^2 \frac{A}{V} - c\right)\right\}}}$.

For the determination of the motion when the solid is not symmetrical with respect to ADB, the reader is referred to Mr Moseley's Hydrostatics (88).

152. Example of the method of finding the height of a mountain by barometrical observations.

According to Gen. Roy (Phil. Trans. 1777), on the 7th of July, 1775, the mean height of the mercury in a barometer on Carnarvon Quay was 30,151 inches, the temperature of the mercury 15°,5, and that of the air 15°,5. On the top of Snowdon the height of the mercury was 26,474 inches, the temperature of the mercury 11°,6, the temperature of the air 9°,5. The latitude of Snowdon is nearly 53°. Here $h = 30{,}151$, $k = 26{,}474$, $s = 15{,}5$, $t = 11{,}6$, $S = 15{,}5$, $T = 9{,}5$, $\lambda = 53°$.

$$\log_{10} h - \log_{10} k = 0{,}0564821, \quad 0{,}000078\,(s - t) = 0{,}0003042,$$

$$120\,(S + T) = 3000, \quad 155 \cos 2\lambda = -43.$$

Hence, (44) if x be the altitude of Snowdon above the Quay in feet, $x = 63302.(0{,}056178) = 3555{,}67$. The value of x determined geometrically was 3555.

153. Two plates of glass meeting in the vertical Cy (fig. 75), and making a very small angle ϵ with each other, are immersed in water; to find the figure of the water elevated between them by capillary attraction.

Let one of the plates meet the surface of the fluid between them in PQ, and the plane of undisturbed surface in Cx. Draw PN parallel to Cy. Let $CN = x$, $PN = y$, the distance between the plates at $P = \epsilon x$; therefore (60) $\epsilon xy = \dfrac{H}{g}$ nearly, the equation to a rectangular hyperbola of which Cx, Cy, are the asymptotes.

154. A wire, the area of a section of which $= \kappa$, can just sustain a weight W without breaking; to find the greatest pressure that can be applied to a fluid contained in a hollow cylinder of the same substance as the wire, without bursting it, a being the radius of the cylinder, and e its thickness.

Let ML (fig. 25) be a portion of the cylinder; MK, HL perpendicular to its axis; MH, KL parallel to its axis; p the pressure of the fluid. The area of the section $MH = e.MH$; therefore it can sustain a tension $\dfrac{e.MH}{\kappa} W$; and (61)

$$p \cdot MH = \frac{e.MH}{\kappa a} W; \quad \therefore\ p = \frac{e}{\kappa a} W.$$

A pressure $\dfrac{2e}{\kappa a} W$ might be applied to a hollow sphere of the same radius and thickness without bursting it.

155. To find the time of emptying a vertical prism or cylinder through a small orifice in its base.

Let A be the area of the base of the prism, κ the area of the orifice, x the depth of the orifice below the surface of the fluid at the end of the time t from the beginning of the motion. Then, (71)

$$\sqrt{(2g)}\,\kappa d_x t = -\frac{A}{\sqrt{x}}; \quad \therefore \sqrt{(2g)}\,\kappa t = C - 2A\sqrt{x};$$

if the depth of the orifice below the surface of the fluid was a when $t = 0$.

$$0 = C - 2A\sqrt{a}; \quad \therefore \sqrt{(2g)}\,\kappa \,.\, t = 2A\,(\sqrt{a} - \sqrt{x});$$

and the whole time of emptying

$$= \frac{A}{\kappa}\sqrt{\left(\frac{2a}{g}\right)}.$$

156. To find the time of emptying a hollow sphere through a small orifice in its vertex.

Let a be the radius of the sphere, κ the area of the orifice, x the depth of the orifice below the surface of the fluid at the end of the time t from the beginning of the motion. The area of the surface of the fluid at the end of the time $t = \pi(2ax - x^2)$;

$$\therefore \sqrt{(2g)}\,\kappa d_x t = -\frac{\pi(2ax - x^2)}{\sqrt{x}} = -\pi(2ax^{\frac{1}{2}} - x^{\frac{3}{2}});$$

$$\therefore \sqrt{(2g)}\,\kappa t = C - \pi\left(\tfrac{4}{3}ax^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right);$$

$$\text{when } t = 0,\ x = 2a; \quad \therefore 0 = C - \tfrac{16}{15}\sqrt{2}\,\pi a^{\frac{5}{2}};$$

$$\therefore \sqrt{(g)}\,\kappa t = \tfrac{16}{15}\pi a^{\frac{5}{2}} - \tfrac{4}{3}\pi a x^{\frac{3}{2}} + \tfrac{2}{5}\pi x^{\frac{5}{2}};$$

and the time of emptying the whole sphere $= \dfrac{16\pi a^{\frac{5}{2}}}{15\kappa\sqrt{(g)}}$.

157. To determine the motion of a fluid oscillating in an inverted syphon PDB (fig. 76) of uniform bore.

Let P, Q be the extremities of the column of fluid at the end of the time t from the beginning of the motion; A, B, the extremities of the column of fluid when at rest; α, β the angles between AP, BQ and the vertical MN; $AP = s$, κ the area of a section of the tube. The moving force on the fluid $= g\rho\kappa \,.\, MN = g\rho\kappa\,(AP\cos\alpha + BQ\cos\beta)$

$$= g\rho\,(\cos\alpha + \cos\beta)\,AP.$$

The mass of the fluid $= \rho\kappa . ADB$, and, since the bore of the tube is uniform, every part of the fluid moves with the same velocity, therefore the effective accelerating force at any point tending to make the fluid return to its position of equilibrium

$$= g(\cos\alpha + \cos\beta)\frac{AP}{ADB}; \quad \therefore d_t^2 s + g(\cos\alpha + \cos\beta)\frac{s}{ADB} = 0;$$

and the time of an oscillation $= \pi\sqrt{\dfrac{ADB}{g(\cos\alpha + \cos\beta)}}$.

158. A weight is raised by a rope wound round the axle of an undershot wheel; to find the velocity of the wheel.

Let K be the area of each float-board; u the velocity of the wheel; v the velocity of the stream; a the radius of the wheel; b the radius of the axle; W the weight. The relative velocity of the stream is $v - u$, and, therefore, the force with which it impels the wheel $= \frac{1}{2}\rho K(v-u)^2$. And when the velocity of the wheel is uniform, this force is in equilibrium with the weight W; $\therefore bW = a\frac{1}{2}\rho K(v-u)^2$.

The work performed by a water wheel is measured by the product of the weight lifted multiplied by the velocity of the weight. (W's velocity) $= W\dfrac{b}{a}u = \frac{1}{2}\rho K(v-u)^2 u$; this is a max. when $3u = v$. The weight lifted in this case $= \dfrac{2}{9}\dfrac{a}{b}\rho K v^2$. The wheel would be kept at rest by a weight $\frac{1}{2}\dfrac{a}{b}\rho K v^2$, therefore the work performed by the wheel is a maximum when the weight lifted is $\frac{4}{9}$ of the weight that would keep the wheel at rest.

159. To find the position of the rudder of a ship, when the effect of the rudder in turning the ship is a maximum.

Let AP (fig. 33) be the keel of the ship, PE perpendicular to the rudder. The resolved part of the resistance on the rudder estimated in a direction perpendicular to AP,

$$\propto (\cos APE)^2 . \sin APE. \quad (86).$$

And this is a maximum when

$$0 = (\cos APE)^3 - 2(\sin APE)^2 . \cos APE, \quad \text{or } \sin APE = \tfrac{1}{3}\sqrt{3}.$$

160. To find the resistance on a sphere.

Let the center of the sphere be the origin of the co-ordinates, a the radius of the sphere, and, therefore, $x^2 + y^2 = a^2$ the equation to its generating circle. Then, $y + x d_y x = 0$,

$$a^2 = x^2 + y^2 = x^2 \{1 + (d_y x)^2\} = (a^2 - y^2)\{1 + (d_y x)^2\};$$

$$\therefore \frac{y}{1 + (d_y x)^2} = y - \frac{y^3}{a^2}, \qquad \int_y \frac{y}{1 + (d_y x)^2} = \frac{y^2}{2} - \frac{y^4}{4a^2} + c,$$

$$\int_y^{\,0\;\;a} \frac{y}{1 + (d_y x)^2} = \frac{a^2}{4},$$

and the resistance on the sphere $= \frac{1}{4}\rho v^2 \pi a^2$.

The resistance on a circular plate, the radius of which is a, $= \frac{1}{2}\rho v^2 \pi a^2$, therefore, by the theory, the resistance on a sphere is half the resistance on a circular plate of the same radius as the sphere. The actual resistance is about one third of the resistance on the circular plate. (Col. Beaufoy's Nautical Experiments, Vol. I. page 481.)

If the density of the sphere $= \sigma$, its mass $= \sigma \frac{4}{3}\pi a^3$ and the retarding force arising from the resistance of the fluid $=$ (resistance) $\div$ (mass of the sphere) $= \frac{3}{16}\frac{\rho}{\sigma}\frac{v^2}{a}$.

161. Example of the determination of the weight of a given volume of water.

A brass sphere appeared to weigh 28704,5 grains when suspended in air, and 49,8 grains when suspended in water; the volume of the sphere at $16^0\frac{2}{3}$ was 113,5264 cubic inches; the temperature of the water $18^0,9$; the temperature of the air $19^0,44$; the altitude of the mercury in the barometer 29,74 inches; the weights were of brass, the density of which is probably about 8 times the density of water. Brass expands 0,0000576 of its volume for one degree of heat, therefore the volume of the sphere at $18^0,9 = 113,5409$ cubic inches. $\text{W} - \text{X} = 28654,7$ grains, therefore, neglecting the small quantities U, W', X', the weight of a cubic inch of water at $18^0,9 = (\text{W} - \text{X}) \div \text{V} = 252\frac{1}{2}$ grains nearly. U = weight of 113,54 cubic inches of air at $19^0,44$, under the pressure of 29,74 inches of mercury at $19^0,44 = 34,17$ grains. $W' - X' =$ weight of air displaced by 28654,7 grains of brass $= \frac{1}{8}(34,47) = 4,27$ grains. Hence, the weight of 113,5407 cubic inches of water at $18^0,9 = 28684,6$ grains, and the weight of one cubic inch $= 252,637$ grains.

162. Example of the comparison of the specific gravities of two fluids.

A glass flask being filled with mercury at 20,6, the mercury appeared to weigh 1340,893 grammes; when filled with water at 20,5, the water appeared to weigh 98,7185 grammes; the weight of the air contained in the flask = 0,1186 grammes; therefore the true weight of the water = 98,8371; and the true weight of the mercury = 1341,0116. The apparent expansion of mercury in glass, between 0^0 and 100^0, = 0,0154, therefore the true weight of the mercury contained in the flask at $20^0,5$

$$= 1341,0116 + (.0000154)(1341) = 1341,0323;$$

$$\therefore \frac{S.G.\text{mercury at } 20^0,5}{S.G.\text{water at } 20^0,5} = \frac{1341,0323}{98,8371} = 13,5681.$$

Between 0^0 and $20^0,5$ the expansion of mercury = 0,00369, and the expansion of water = 0,001698;

$$\therefore \frac{S.G.\text{mercury at } 0^0}{S.G.\text{water at } 0^0} = 13,5681\,\frac{1,00369}{1,001698} = 13,5952.$$

163. The pressure of a fluid is frequently expressed in "atmospheres," an atmosphere denoting the pressure of a column of mercury at $0^0 C$, 0,76 mètres, or 29,9218 inches high, at the mean level of the sea in lat. 45^0. At the mean level of the sea in lat. λ, an atmosphere is the pressure of a column of mercury at $0^0 C$, $0,76 + 0,001946 \cos 2\lambda$ mètres, or $29,9218 + 0,0766 \cos 2\lambda$ inches high.

If T be the temperature of steam, Υ its pressure in atmospheres, it is found that up to 224^0, and probably much higher,

$$\text{T} = 100 + 64,29512\,(\log_{10}\Upsilon) + 13,89479\,(\log_{10}\Upsilon)^2$$
$$+ 2,909769\,(\log_{10}\Upsilon)^3 + 0,1742634\,(\log_{10}\Upsilon)^4.$$

$\log_{10} 64,29512 = 1,8081780$, $\log_{10} 13,89479 = 1,1428520$,

$\log_{10} 2,909769 = 0,4638586$, $\log_{10} 0,1742634 = \bar{1},2412062$.

Pressure of vapour at different temperatures from -20^0 to 35^0, in inches of mercury, deduced from the observations of Kämtz:

-20^0	0,040	6^0	0,272	16^0	0,519	26^0	0,949
-15	0,060	7	0,290	17	0,553	27	1,006
-10	0,087	8	0,311	18	0,588	28	1,066
-5	0,126	9	0,332	19	0,625	29	1,029
0	0,180	10	0,354	20	0,664	30	1,195
1	0,194	11	0,378	21	0,705	31	1,265
2	0,207	12	0,403	22	0,750	32	1,338
3	0,222	13	0,430	23	0,796	33	1,415
4	0,238	14	0,457	24	0,844	34	1,496
5	0,254	15	0,488	25	0,895	35	1,580

164. Ratios of the specific gravities of different substances to that of water at $15^0,5\,C$, or $60^0 F$:

Diamond	3,52	Uranium	9,
Sulphur	2,	Bismuth	9,83
Phosphorus	1,75	Tin	7,285
Iodine	4,94	Lead	11,445
Natrium	0,972	Cadmium	8,604
Kalium	0,865	Zinc	6,862
Selenium	4,32	Nickel	8,38
Arsenic	5,959	Cobalt	8,513
Chrome	5,9	Iron	7,844
Molybdenum	8,63	Manganese	8,013
Tungsten	17,6		
Antimony	6,86	Flint Glass	3,33
Tellurium	6,2578	Plate Glass	2,5
Titanium	5,3	Marble	2,716
Gold	19,4	Quartz	2,6
Osmium	10,	Rock Salt	1,92
Iridium	18,68	Ivory	1,917
Platina	21,53	Ice (at 0^0)	0,926
Palladium	11,3	Sea Water	1,027
Rhodium	11,2	Olive Oil	0,915
Silver	10,5	Alcohol	0,7947
Mercury	13,568	Naphta	0,753
Copper	8,85	Æther	0,724

165. Hällström infers from his own experiments, combined with those of Muncke and Stampfer, that the density of water is a maximum at 3,92, and that between 0° and 30° the volume of a given mass of water at t° divided by its volume at 0°

$$= 1 - 0{,}000057577t + 0{,}0000075601t^2 - 0{,}000000035091t^3,$$

and between 30° and 100°

$$= 1 - 0{,}0000094178t + 0{,}000005333661t^2 - 0{,}0000000104086t^3.$$

From these two expressions the following table of the volume and density of water at different temperatures has been computed :

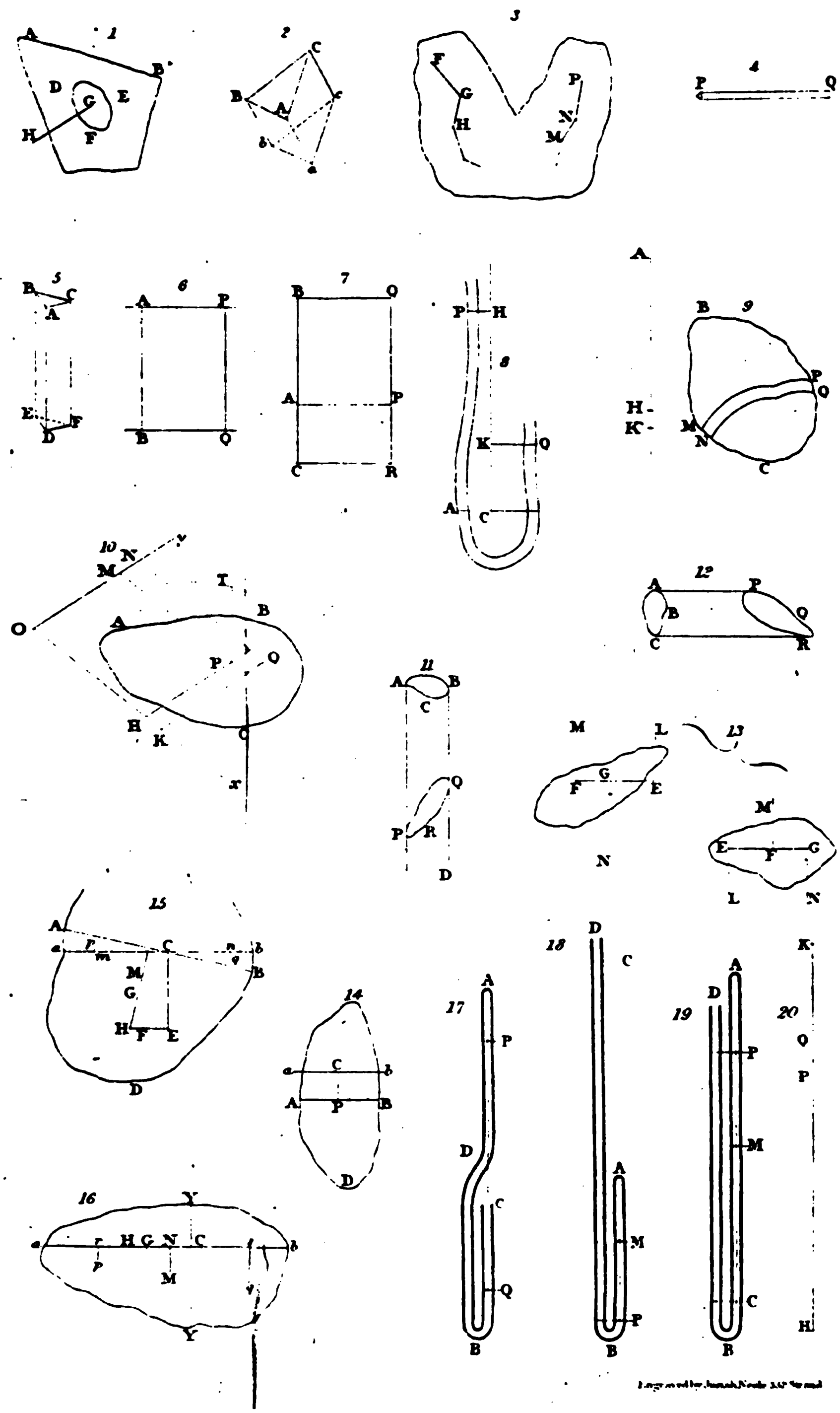
Temp.	Volume.	Density.	Temp.	Volume.	Density.
0	1,000000	1,000000	22	1,002022	0,997982
1	0,999950	1,000050	23	1,002251	0,997754
2	0,999915	1,000085	24	1,002491	0,997515
3	0,999894	1,000106	25	1,002741	0,997267
3,9	0,999882	1,000118	26	1,003001	0,997008
4	0,999888	1,000112	27	1,003271	0,996740
5	0,999897	1,000103	28	1,003549	0,996463
6	0,999919	1,000081	29	1,003837	0,996178
7	0,999956	1,000044	30	1,004216	0,995802
8	1,000006	0,999994	35	1,005761	0,994272
9	1,000069	0,999931	40	1,007496	0,992560
10	1,000145	0,999855	45	1,009434	0,990654
11	1,000235	0,999765	50	1,011570	0,988563
12	1,000338	0,999662	55	1,013894	0,986297
13	1,000453	0,999547	60	1,016398	0,983867
14	1,000581	0,999419	65	1,019078	0,981280
15	1,000720	0,999280	70	1,021920	0,978550
16	1,000872	0,999128	75	1,024921	0,975685
17	1,001035	0,998966	80	1,028072	0,972695
18	1,001210	0,998791	85	1,031364	0,969590
19	1,001397	0,998605	90	1,034791	0,966379
20	1,001594	0,998408	95	1,038346	0,963070
21	1,001802	0,998201	100	1,042016	0,959678

176. Ratios of the densities of gases and vapours of different substances to that of atmospheric air, at the same temperature and under the same pressure.

Oxygen	1,1026	Nitric Oxide	1,0393
Hydrogen	0,0688	Hyponitrous Acid	1,5906
Nitrogen	0,976	Ammonia	0,5912
Chlorine	2,44033	Hydrochloric Acid	1,2544
Bromine	5,395	Sulphurous Acid	2,21162
Iodine	8,70111	Sulphuric Acid	2,76292
Sulphur	6,654	Arsenious Acid	13,3
Phosphorus	4,326	Naphta	2,96
Arsenic	10,36536	Æther	2,586
Mercury	6,97848	Protochl. of Mercury	8,2
Water	0,6201	Chloride of Mercury	9,42
Alcohol	1,6133	Cyanogen	1,81879
Carbonic Acid	1,5245	Hydrocyanic Acid	0,94379
Nitrous Oxide	1,5273	Carbonic Acid	1,5245

According to the experiments of Dr Prout, 100 cubic inches of dry atmospheric air, free from carbonic acid, at $0^\circ C$, under the pressure of 30 inches of mercury at $0^\circ C$, in the latitude of London, weigh 32,7958 grains; and 100 cubic inches of the same air at $15^\circ,55\,C$ ($60^\circ F$), under the same pressure, weigh 31,0117 grains. According to the observations of Dumas and Boussingault, the density of dry air at 0°, under the pressure of one atmosphere, divided by the maximum density of water = 0,0012995.

Plate I.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2
C B A c b a
3
F G H P N M
4
P Q
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B C A E D F
6
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7
B Q A P C R
8
P H K Q A C
9
A B H K M N P Q C
10
N M v O T A B P Q H K C x
11
A B C Q P R D
12
A P B Q C R
13
M L G F E N M' E F G L N
15
A a C b B M G H F E D
14
a C b A P B D
16
Y a H G N C b M Y
17
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18
D C A M P B
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A D P M C B
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K Q P H

Plate 2.

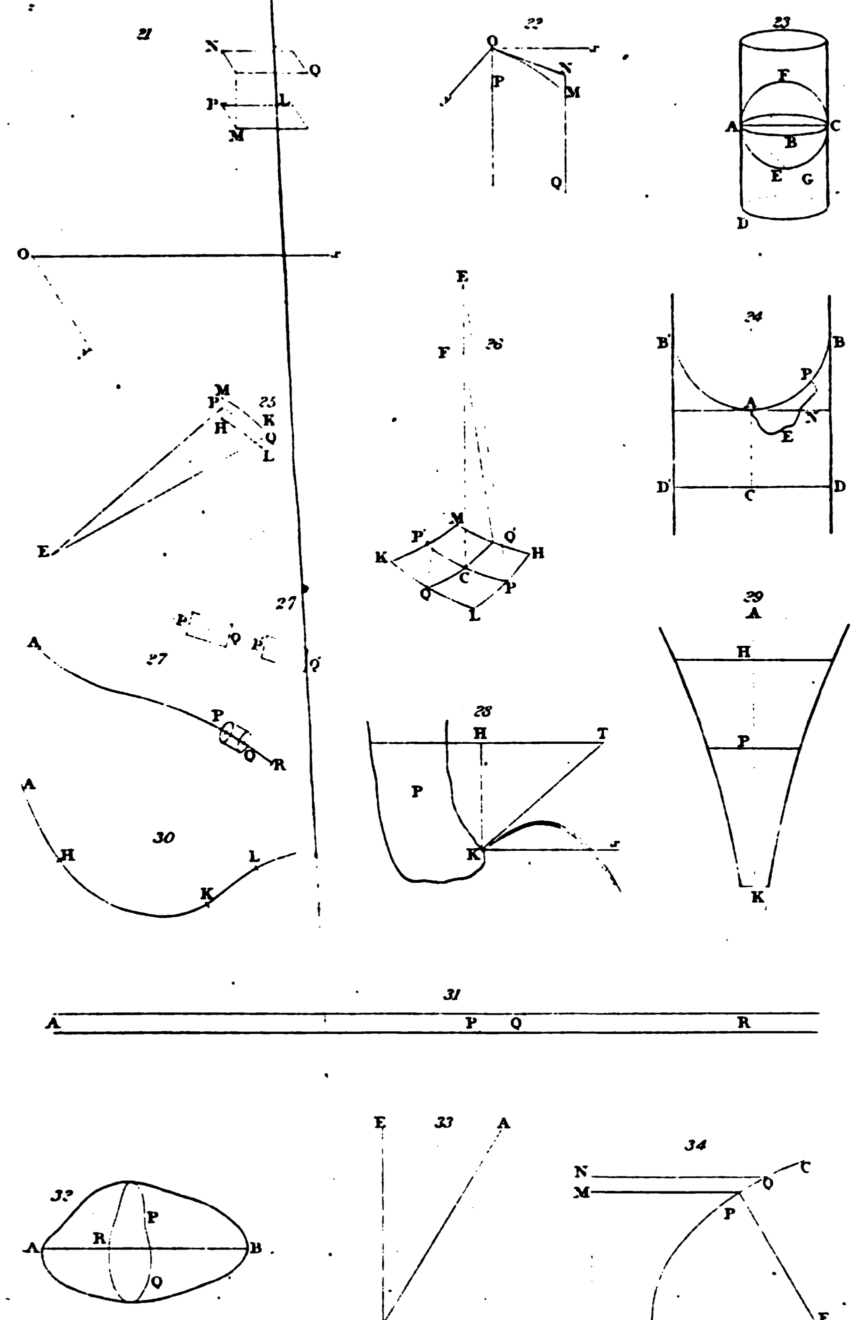

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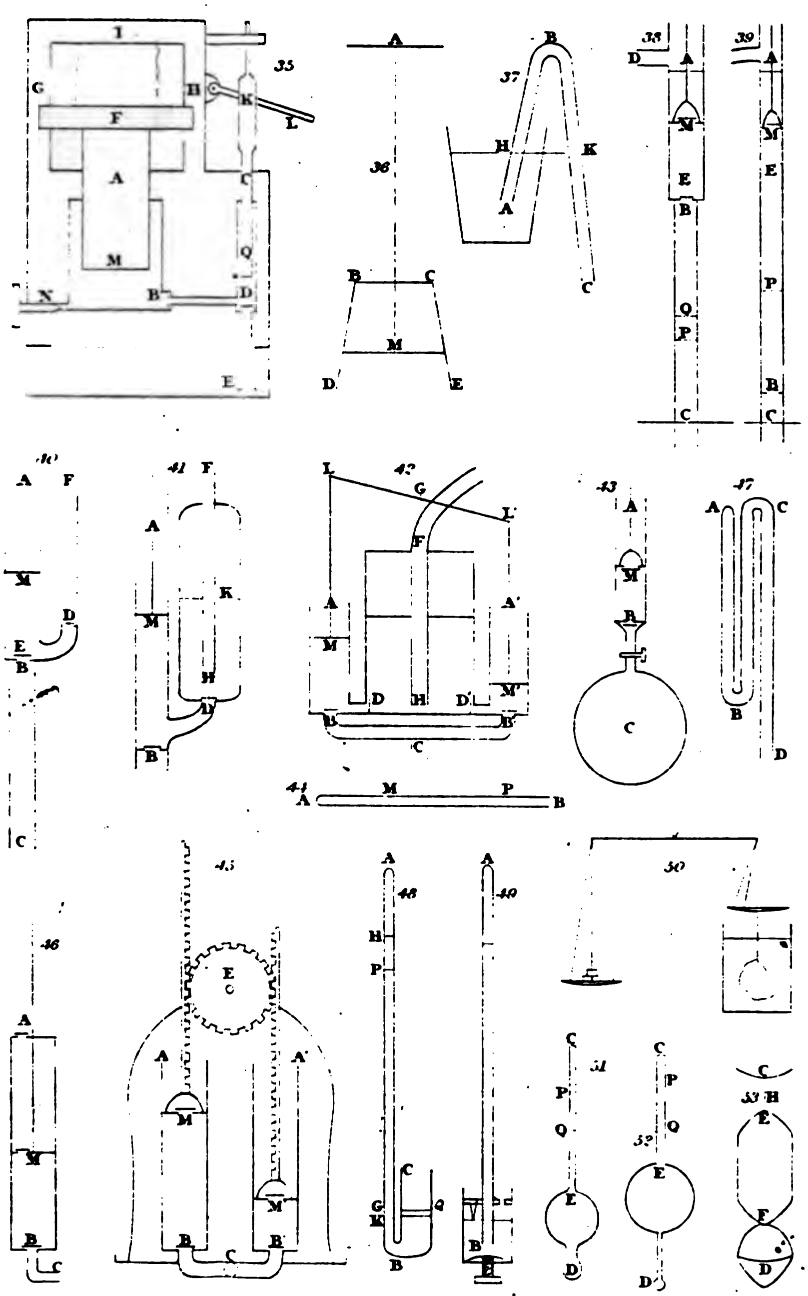
Plate 3.



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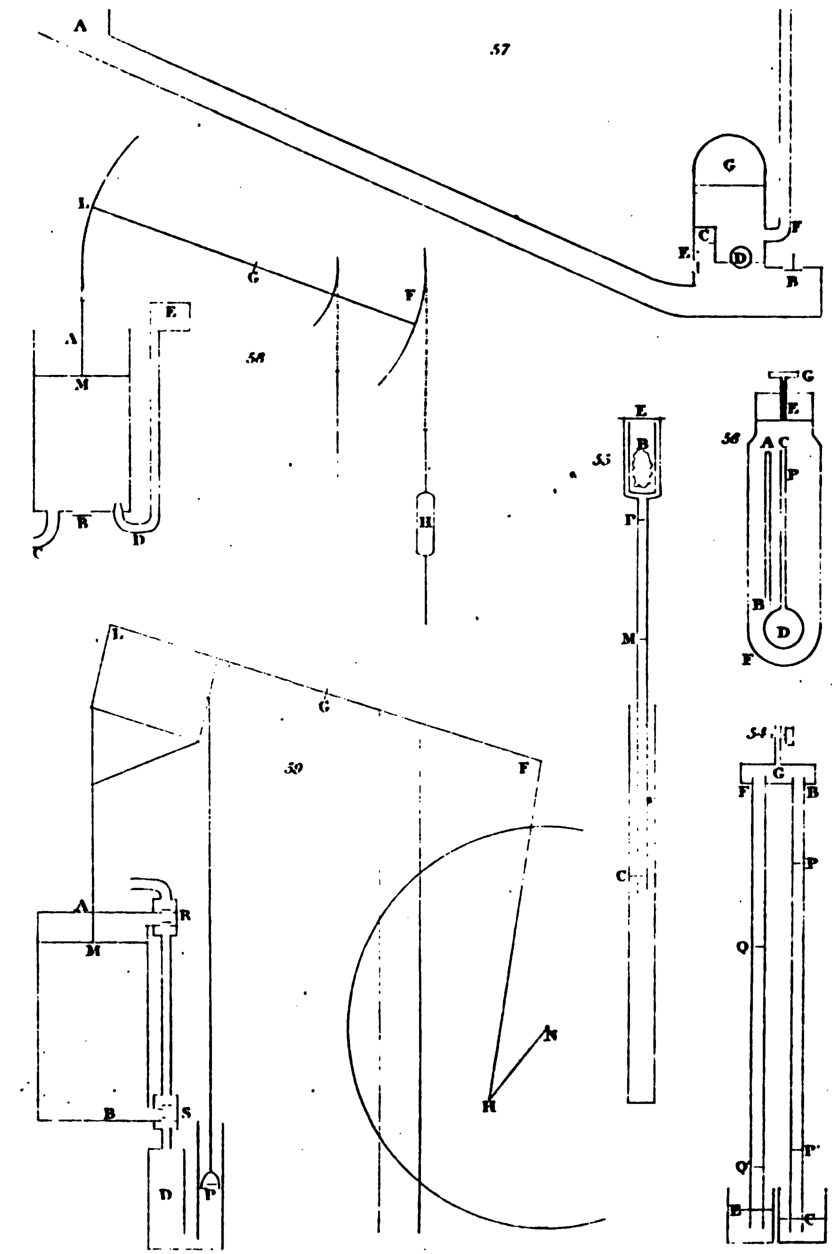

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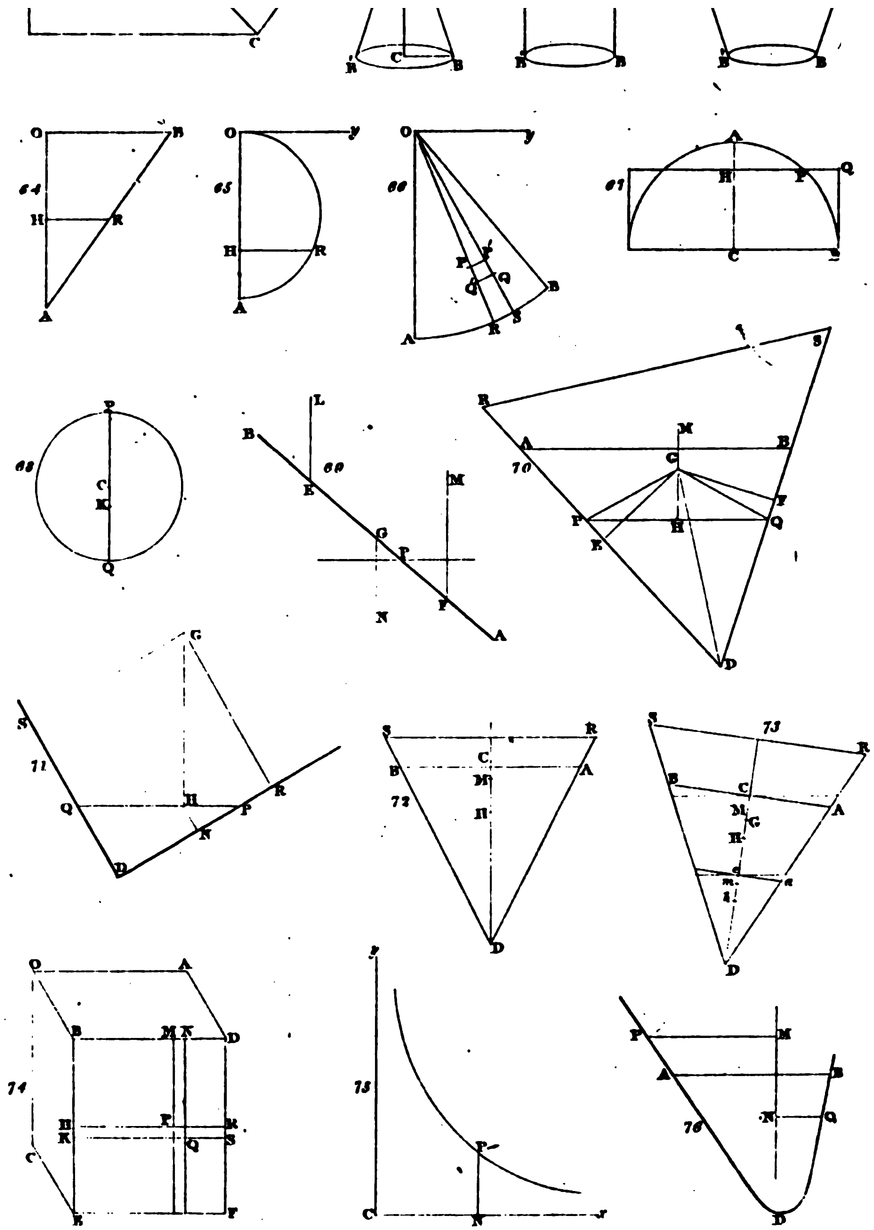

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